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DEPARTMENT OF PHYSICS

September 27, 1962

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Dear Feza:

Thank you for your letter of August 26 which contains several very interesting remarks. Enclosed please find a copy of my notes. I hope you will excuse me from sending you a bulk of materials that have been neither edited nor organized.

At first sight, it might appear that there are three different formulations of spin $\frac{1}{2}$ equation in de Sitter space; namely, the Dirac's formulation (Eq. (27) of the notes), the conventional formulation with covariant derivative (Eq. (38)) and the variational method with constrain (page 8 of the notes). It is shown in the notes that all three formulations are, in fact, identical. (See § 6 and § 7.) For m = 0 they all admit 2-component solution.

Indeed, if not because of the remarks that you made in your last letter, I would not have even suspected that Dirac's formulation is identical with the other two methods. I would like very much to hear your conclusions, and I hope you will find these notes interesting and not too unclear.=

It is very nice to hear from you. All of us are looking forward to our visit to Turkey next year. Please just put us in a hotel.

At present, I am living a commuter's life. It takes about $3\frac{1}{2}$ hours to go to New York and back to Princeton each day. In spite of the obvious disadvantage, it does give me an opportunity to catch up on my reading.

With my very best regards to you and Suha, and hoping to hear from you soon, I am

Yours sincerely,

Tsung-Dao
Tsung-Dao (T.D. Lee)

TDL:jd
Enclosure

P.S. I am looking now at the improper transformations in de-Sitter space. ~~There seems to be~~ It seems rather interesting. Will write to you soon. T.D.

Spin $\frac{1}{2}$ Eqn in de Sitter Space.

§1. Some Definitions and Some Identities.

1. All Greek super- and sub-scipts, such as

$$\alpha, \beta, \cdots, \mu, \nu, \cdots ,$$

vary from 1 to 5.

2. All Roman super- and sub-scipts, such as

$$a, b, \cdots, i, j, k, \cdots$$

vary from 1 to 4.

3. ξ^μ are all <u>real</u>

4. De Sitter Space:

$$\xi^\mu \xi^\nu \eta_{\mu\nu} = \text{constant} \equiv R^2 \qquad (1)$$

where

$$\eta \equiv (\eta_{\mu\nu}) = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & -1 & \\ & & & & 1 \end{pmatrix} \qquad (2)$$

5. $\xi_\mu \equiv \eta_{\mu\nu} \xi^\nu \qquad (3)$

and $\eta^{\mu\nu} = \eta_{\mu\nu}$

$\therefore \quad \xi^\mu = \eta^{\mu\nu} \xi_\nu$

6. γ^μ = constant (4×4) matrix. $\gamma_\mu \equiv \eta_{\mu\nu}\gamma^\nu$

$$\underline{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}} \quad ; \qquad (4)$$

7. Rotation Operators : $\underline{J_{\mu\nu} = L_{\mu\nu} + \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)}$ (5)

$$\underline{L_{\mu\nu} \equiv \xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}} \qquad (5)$$

8. It is useful to transform

$$\xi^\mu \rightarrow x^\alpha$$

where $\underline{x^5 \equiv \sqrt{\xi^\mu \xi^\nu \eta_{\mu\nu}} \equiv r}$ (6)

and is orthogonal to x^i $(i = 1, 2, 3, 4)$

i.e. $\underline{\left(\frac{\partial \xi^\mu}{\partial r}\right)\left(\frac{\partial \xi^\nu}{\partial x^i}\right)\eta_{\mu\nu} = 0}$ (7)

The x^i's are otherwise <u>completely arbitrary</u>. [cf. p.5]

9. $$ds^2 = \eta_{\mu\nu}\, d\xi^\mu d\xi^\nu = g_{ij}\, dx^i dx^j + dr^2 \qquad (8)$$

$$= g_{\mu\nu}\, dx^\mu dx^\nu$$

where $$g_{ij} = \frac{\partial \xi^\mu}{\partial x^i}\frac{\partial \xi^\nu}{\partial x^j}\eta_{\mu\nu} \quad \& \quad (g_{\mu\nu}) = \left(\begin{array}{c|c} g_{ij} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline 0\,0\,0\,0 & 1 \end{array}\right) \qquad (9)$$

Identities

1. $$\left(\frac{\partial \xi^\mu}{\partial r}\right)_{x^i} = \eta^{\mu\nu}\left(\frac{\partial r}{\partial \xi^\nu}\right)_{\xi^{\nu\neq\mu}} = \frac{\xi^\mu}{r} \qquad \text{(indep. of } r\text{)} \qquad (10)$$

Pf. Define $A_{\mu\nu} \equiv \frac{\partial \xi^\mu}{\partial x^\nu}$ $\qquad (A^{-1})_{\mu\nu} = \frac{\partial x^\mu}{\partial \xi^\nu}$

$$\tilde{A}\,\eta\,A = \left(\begin{array}{c|c} g_{ij} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline 0\;0\;0\;0 & 1 \end{array}\right) \equiv g$$

$$\therefore \quad A = \eta\,\tilde{A}^{-1} g \qquad \text{i.e. } A_{\mu 5} = \eta_{\mu\nu}(\tilde{A}^{-1})_{\nu 5}$$

$$= \eta_{\mu\nu}\frac{\partial r}{\partial \xi^\nu} = \frac{\xi^\mu}{r}$$

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2. $$\Gamma^\mu_{\nu\sigma} \equiv \frac{1}{2} g^{\mu\alpha}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\alpha}\right) = \text{Christoffel Symbol}$$

then $$\Gamma^\mu_{\nu\sigma} = \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\sigma} = -\frac{\partial \xi^\alpha}{\partial x^\sigma}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^\mu}{\partial \xi^\alpha}\right) \qquad (11)$$

Pf. $$\frac{\partial}{\partial x^\sigma}(g_{\alpha\nu}) = \frac{\partial}{\partial x^\sigma}\left(\frac{\partial \xi^\lambda}{\partial x^\alpha}\frac{\partial \xi^\varkappa}{\partial x^\nu}\eta_{\lambda\varkappa}\right)$$

$$\frac{\partial}{\partial x^\nu}(g_{\alpha\sigma}) = \frac{\partial}{\partial x^\nu}\left(\frac{\partial \xi^\varkappa}{\partial x^\alpha}\frac{\partial \xi^\lambda}{\partial x^\nu}\eta_{\lambda\varkappa}\right)$$

$$-\frac{\partial}{\partial x^\alpha}(g_{\nu\sigma}) = -\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \xi^\varkappa}{\partial x^\nu}\frac{\partial \xi^\lambda}{\partial x^\sigma}\eta_{\varkappa\lambda}\right)$$

$$\text{Sum} = 2\,\frac{\partial \xi^\lambda}{\partial x^\alpha}\frac{\partial^2 \xi^\varkappa}{\partial x^\sigma \partial x^\nu}\eta_{\varkappa\lambda}$$

$$g^{\mu\alpha} = \frac{\partial x^\mu}{\partial \xi^{\varkappa'}}\frac{\partial x^\alpha}{\partial \xi^{\lambda'}}\eta^{\lambda'\varkappa'}$$

$\therefore$ Q.E.D.

3. $\Gamma^{i}_{i5} = \frac{4}{r}$

$$\Gamma^{5}_{5\sigma} = \Gamma^{\mu}_{55} = \Gamma^{5}_{\sigma 5} = 0 \qquad (12)$$

Pf. $\Gamma^{5}_{5\sigma} = -\frac{\partial \xi^\alpha}{\partial x^\sigma} \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial \xi^\alpha}\right) = -\frac{\partial \xi^\alpha}{\partial x^\sigma}\frac{\partial}{\partial r}\left(\frac{\xi_\alpha}{r}\right) = 0$

$\Gamma^{\mu}_{55} = \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial}{\partial r}\left(\frac{\xi^\alpha}{r}\right) = 0$

$\Gamma^{i}_{i5} = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial}{\partial x^i}\left(\frac{\partial \xi^\alpha}{\partial x^5}\right) = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial}{\partial x^i}\left(\frac{\xi^\alpha}{x^5}\right) = \frac{4}{r}$.

4. Commutators :

$$\left[\gamma^\mu \frac{\partial}{\partial \xi^\mu}, \gamma^\alpha \xi_\alpha\right] = 5 - \gamma^\mu\gamma^\nu L_{\mu\nu} \qquad (13)$$

$$\left[\gamma^\mu\gamma^\nu L_{\mu\nu}, \gamma^\alpha\xi_\alpha\right] = 4\gamma^\alpha\xi_\alpha\, r\frac{\partial}{\partial r} - 4r^2\gamma^\mu\frac{\partial}{\partial \xi^\mu} + 8\gamma^\mu\xi_\mu \qquad (14)$$

where $\frac{\partial}{\partial r} = \frac{\xi^\mu}{r}\frac{\partial}{\partial \xi^\mu}$

5. $$\gamma^\mu\gamma^\nu L_{\mu\nu} = 2(\gamma^\mu\xi_\mu)\left(\gamma^\alpha\frac{\partial}{\partial \xi^\alpha}\right) - 2r\frac{\partial}{\partial r} \qquad (15)$$

$$(\gamma^\alpha\xi_\alpha)(\gamma^\mu\gamma^\nu L_{\mu\nu}) = 2r^2\gamma^\mu\frac{\partial}{\partial\xi^\mu} - 2(\gamma^\alpha\xi_\alpha)\, r\frac{\partial}{\partial r}$$

& $$(\gamma^\mu\gamma^\nu L_{\mu\nu})(\gamma^\alpha\xi_\alpha) = -(\gamma^\alpha\xi_\alpha)(\gamma^\mu\gamma^\nu L_{\mu\nu}) + 8\gamma^\alpha\xi_\alpha \qquad (16)$$

Pf. $(13)_{\text{left-hand side}} = \gamma^\mu\gamma^\nu\left(\frac{\partial}{\partial\xi^\mu}\xi^\nu - \xi^\mu\frac{\partial}{\partial\xi^\nu}\right) = -\gamma^\mu\gamma^\nu L_{\mu\nu} + \gamma^\mu\gamma^\nu\eta_{\mu\nu}$.

$L_{\mu\nu}\xi_\alpha = \xi_\alpha L_{\mu\nu} + \xi_\mu\eta_{\nu\alpha} - \xi_\nu\eta_{\mu\alpha}$

$\gamma^\mu\gamma^\nu\gamma^\alpha = \gamma^\alpha\gamma^\mu\gamma^\nu + 2\eta^{\alpha\nu}\gamma^\mu - 2\eta^{\alpha\mu}\gamma^\nu$

$\therefore\ (14)_{\text{Left-hand}} = \gamma_\nu\gamma^\mu\gamma^\nu\xi_\mu - 5\gamma^\nu\xi_\nu + 2\gamma^\mu\xi^\nu L_{\mu\nu} - 2\gamma^\nu\xi^\mu L_{\mu\nu}$
$+ (10\gamma^\mu\xi_\mu - 2\gamma^\mu\xi_\mu)2$

$= \gamma^\mu\xi_\mu(-10 + 2 + 20 - 4) + (2+2)\left[(\gamma^\mu\xi_\mu)\left(\xi^\nu\frac{\partial}{\partial\xi^\nu}\right) - r^2\gamma^\mu\frac{\partial}{\partial\xi^\mu}\right]$

$\gamma^\mu\gamma^\nu L_{\mu\nu} = 2(\gamma^\mu\xi_\mu)\left(\gamma^\nu\frac{\partial}{\partial\xi^\nu}\right) - 2\eta^{\mu\nu}\xi_\mu\frac{\partial}{\partial\xi^\nu}$ gives (15)

Choice of Curvilinear System.

Let $$\xi^i = \frac{r}{R} f^i(x^1, x^2, x^3, x^4) \qquad i = 1, 2, 3, 4$$

$$\xi^5 = \sqrt{r^2 - \xi^i \xi_i / R^2} = r \sqrt{1 - \frac{1}{R^2} f^i f_i} \qquad (17)$$

then

$$r^2 = \xi^\mu \xi_\mu$$

$$\frac{\partial r}{\partial \xi^\mu} = \frac{\xi_\mu}{r} \qquad \frac{\partial \xi^\mu}{\partial r} = \frac{\xi^\mu}{r}$$

$$\frac{\partial \xi^\mu}{\partial r} \frac{\partial \xi^\nu}{\partial x^i} \eta_{\mu\nu} = \frac{1}{r} \xi^\mu \frac{\partial \xi^\nu}{\partial x^i} \eta_{\mu\nu} = \frac{1}{r} \frac{\partial}{\partial x^i} \left(\xi^\mu \xi^\nu \eta_{\mu\nu} \right) \frac{1}{2}$$

$$= 0 \, .$$

*

Def^n

$$g_0 \equiv \det \left| \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} \eta_{ab} \right| = - \left| \frac{\partial f^i}{\partial x^j} \right|^2$$

$$g \equiv \det \left| \frac{\partial \xi^a}{\partial x^i} \frac{\partial \xi^b}{\partial x^j} \eta_{ab} \right| = \left(\frac{r}{R} \right)^8 g_0 \qquad (18)$$

Notice that 1. $g_0 = g_0(x^1, x^2, x^3, x^4)$ only

~~Notice that~~ 2. the choice of f^i is completely arbitrary.

Important Identity:

$$\frac{\partial}{\partial x^i} \left(\sqrt{-g_0} \, \frac{\partial x^i}{\partial \xi^\mu} \right) = - \frac{4}{r^2} \xi_\mu \sqrt{-g_0} \qquad (19)$$

Pf. $$\frac{\partial\sqrt{-g_0}}{\partial x^i} = \sqrt{-g_0}\,\Gamma^{j}_{j\,i}$$

by (11)

and $$\frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial \xi^\mu}\right) = -\frac{\partial x^\sigma}{\partial \xi^\mu}\Gamma^{i}_{i\,\sigma} = -\frac{\partial x^j}{\partial \xi^\mu}\Gamma^{i}_{i\,j} - \frac{\partial x^5}{\partial \xi^\mu}\Gamma^{i}_{i\,5}$$

Hence $$\frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial \xi^\mu}\sqrt{-g_0}\right) = -\Gamma^{i}_{i\,5}\frac{\partial x^5}{\partial \xi^\mu}\sqrt{-g_0} = -\frac{4\,\xi_\mu}{r^2}\sqrt{-g_0}$$

by (12)

Alternative Proof

$\because \frac{\partial\sqrt{-g}}{\partial x^\mu} = \Gamma^{\nu}_{\nu\mu}\sqrt{g}$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^\nu}{\partial \xi^\mu}\sqrt{-g}\right) = 0$$

fn of x^i only

$$\parallel$$

$$\frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial \xi^\mu}\sqrt{-g}\right) + \frac{\partial}{\partial x^5}\left[\frac{\xi_\mu}{x^5}\cdot\left(\frac{x^5}{R}\right)^4\sqrt{-g_0}\right]$$

$$\therefore\quad \underline{\frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial \xi^\mu}\sqrt{-g}\right) = -\frac{4\,\xi_\mu}{r^2}\sqrt{-g}}$$

§2. Variation Principle – Fundamental Eqn

Defn.

This is used, instead of $(\bar{\psi}\gamma^\mu \frac{\partial}{\partial\xi^\mu}\psi)$, to maintain hermiticity

$$\mathcal{L} = i \int_D \left[\frac{1}{2} \left(\bar{\psi} \gamma^\mu \frac{\partial \psi}{\partial \xi^\mu} - \frac{\partial \bar{\psi}}{\partial \xi^\mu} \gamma^\mu \psi \right) - m \bar{\psi} \psi \right] \prod_1^5 d\xi^\mu \qquad (20)$$

D = region bounded by $r = R$ & $r = R + \epsilon$

$\epsilon \to 0+$

$\bar{\psi} = \psi^\dagger \gamma^4$

Simple Properties :

1. $\mathcal{L}$ = invariant under arb. rotation in de Sitter space

2. $(\gamma^4)^\dagger = -\gamma^4$ & $(\gamma^\mu)^\dagger = +\gamma^\mu$ $\quad \mu \neq 4$

$(\gamma^\alpha)^\dagger \gamma^4 = -\gamma^4 \gamma^\alpha \qquad \therefore \quad (\gamma^4\gamma^\mu)^\dagger = +\gamma^4\gamma^\mu$

3. $\mathcal{L}^* = \mathcal{L}$ $\quad$ ($\psi \neq 0$ on the surface S_D of D) $\quad$ (21)

$$\text{Pf.} \left[i \int_D \left(\bar{\psi} \gamma^\mu \frac{\partial \psi}{\partial \xi^\mu} - \frac{\partial \bar{\psi}}{\partial \xi^\mu} \gamma^\mu \psi \right) d^5\xi \right]^* = -i \int_D \left[\tilde{\psi} (\gamma^4\gamma^\mu)^* \frac{\partial \psi^*}{\partial \xi^\mu} - \frac{\partial \tilde{\psi}}{\partial \xi^\mu} (\gamma^4\gamma^\mu)^* \cdot \psi^* \right] d^5\xi$$

$$= -i \int_{S_D} \left[\tilde{\psi} (\gamma^4\gamma^\mu)^* \psi^* - \tilde{\psi} (\gamma^4\gamma^\mu)^* \psi^* \right] dS_\mu$$

$$+ i \int_D \left[\psi^\dagger (\gamma^4\gamma^\mu)^\dagger \frac{\partial \psi}{\partial \xi^\mu} - \frac{\partial \psi^\dagger}{\partial \xi^\mu} (\gamma^4\gamma^\mu)^\dagger \psi \right] d^5\xi$$

$$= \left[i \int \left(\bar{\psi} \gamma^\mu \frac{\partial \psi}{\partial \xi^\mu} - \frac{\partial \bar{\psi}}{\partial \xi^\mu} \gamma^\mu \psi \right) d^5\xi . \right.$$

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The eq'n given in my last letter is <u>incorrect</u>!

Variation Principle

$$\delta \mathcal{L} = 0 \qquad \text{for arb. } \delta\psi \quad \text{where}$$

$$\psi = \text{indep. of } r$$

$i = 1,2,3,4$ only

$\mu = 1,2,3,4,5$

Theorem 1.

$$\underline{M^i \frac{\partial \psi}{\partial x^i} - \frac{2}{R} M^5 \psi - m\psi = 0} \qquad (22)$$

indep of r

where $$M^\alpha = \frac{\partial x^\alpha}{\partial \xi^\mu}\gamma^\mu \qquad \left(\therefore \quad M^5 = \frac{\xi_\mu}{r}\gamma^\mu\right) \qquad (23)$$

Pf. $$\frac{i}{2}\int_D \left[\delta\bar\psi\left(\gamma^\mu \frac{\partial}{\partial \xi^\mu}\psi\right) + \bar\psi\gamma^\mu\frac{\partial}{\partial\xi^\mu}\delta\psi - \left(\frac{\partial}{\partial\xi^\mu}\delta\bar\psi\right)\gamma^\mu\psi - \frac{\partial\bar\psi}{\partial\xi^\mu}\gamma^\mu\delta\psi\right] d^5\xi \qquad (24)$$

Let $$\gamma^\mu \frac{\partial}{\partial\xi^\mu} = M^i\frac{\partial}{\partial x^i} + M^5\frac{\partial}{\partial r}$$

$$d^5\xi = (d^4x)\left(\frac{r}{R}\right)^4\sqrt{-g_0}\, dr$$

$$\frac{\partial\psi}{\partial r} = \frac{\partial\,\delta\psi}{\partial r} = 0$$

(24) becomes

$$\frac{i}{2}\int_D \left[\delta\bar\psi\left(M^i\frac{\partial}{\partial x^i}\psi\right) - \frac{\partial\bar\psi}{\partial x^i}M^i\delta\psi\right] d^4x \left(\frac{r}{R}\right)^4\sqrt{-g_0}\, dr$$

$$+ \frac{i}{2}\int\left(\frac{r}{R}\right)^4 dr\left\{\delta\bar\psi\frac{\partial}{\partial x^i}(M^i\sqrt{-g_0}\,\psi) - \left[\frac{\partial}{\partial x^i}(\bar\psi M^i\sqrt{-g_0})\right]\delta\psi\right\} d^4x \qquad (25)$$

By using (19)

$$\frac{\partial}{\partial x^i}(M^i\sqrt{-g_0}) = -\frac{4M^5}{r}\sqrt{g_0} \qquad (26)$$

$\therefore$

$$(24) = i \int \left\{ \delta\bar{\psi} \left[M^i \frac{\partial \psi}{\partial x^i} - \frac{2 M^5}{r} \psi \right] - \left[\frac{\partial \bar{\psi}}{\partial x^i} M^i - \bar{\psi} \frac{2 M^5}{r} \right] \delta\psi \right\} \sqrt{-g}\, d^4$$

$$\cdot \left(\frac{r}{R}\right)^4 dr$$

$\therefore$

$$M^i \frac{\partial \psi}{\partial x^i} - 2 \frac{M^5}{r} \psi - m\psi = 0$$

$$\& \quad + \frac{\partial \bar{\psi}}{\partial x^i} M^i - \frac{2 \bar{\psi}}{r} M^5 + m\bar{\psi} = 0$$

<u>Remarks</u> We may try $\psi = \dfrac{\phi}{r^a}$ (a = const.)

& ϕ = indep. of r

$$M^5 \frac{\partial \psi}{\partial r} = - \frac{a}{r} M^5 \psi \quad ;$$

$$\therefore \quad (24) = (25) + \frac{i}{2} \int\!\!\int \left[\delta\bar{\psi} \left(- \frac{a}{r} M^5 \psi \right) + \bar{\psi} \left(- \frac{a}{r} M^5 \delta\psi \right) \right.$$

$$\left. - \left(- \frac{a}{r} \delta\bar{\psi} M^5 \psi \right) - \left(- \frac{a}{r} \bar{\psi} M^5 \delta\psi \right) \right] d^5\xi$$

$$= (25)$$

$\therefore$ Same result !

§3. Rotation Invariance.

That (22) is invariant under arb. 5 dim. rotation is obvious from the invariant character of the variation principle.

We give an alternative (but more explicit) proof:

<u>Theorem 2</u>.

$$[J_{\mu\nu}, r] = 0$$

$$[J_{\mu\nu}, M^5] = 0$$

$$[J_{\mu\nu}, M^i \frac{\partial}{\partial x^i}] = 0 \qquad (27)$$

<u>Pf</u>. $J_{\mu\nu} = L_{\mu\nu} + \frac{1}{2}\gamma_\mu\gamma_\nu$ & $L_{\mu\nu} = \xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}$

$$[L_{\mu\nu}, r] = \frac{1}{r}(\xi_\mu \xi_\nu - \xi_\nu \xi_\mu) = 0$$

$$[L_{\mu\nu}, \xi^\alpha] = \delta^\alpha_\nu \xi_\mu - \delta^\alpha_\mu \xi_\nu$$

$$[L_{\mu\nu}, \frac{\partial}{\partial \xi^\alpha}] = -\eta_{\alpha\mu} \frac{\partial}{\partial \xi^\nu} + \eta_{\alpha\nu} \frac{\partial}{\partial \xi^\mu}$$

$$\frac{1}{2}[\gamma_\mu\gamma_\nu, \gamma^\alpha] = \delta^\alpha_\nu \gamma_\mu - \delta^\alpha_\mu \gamma_\nu$$

$$\therefore \quad [J_{\mu\nu}, \gamma^\alpha \frac{\partial}{\partial \xi^\alpha}] = [J_{\mu\nu}, \overset{= r\frac{\partial}{\partial r}}{\xi^\alpha \frac{\partial}{\partial \xi^\alpha}}] = [J_{\mu\nu}, \overset{= M^5 r}{\gamma^\alpha \xi_\alpha}] = 0$$

$$M^i \frac{\partial}{\partial x^i} = \gamma^\alpha \frac{\partial}{\partial \xi^\alpha} - M^5 \frac{\partial}{\partial r} \qquad \therefore \quad [J_{\mu\nu}, M^i \frac{\partial}{\partial x^i}] = 0$$

§ 4. Current Conservation

<u>Theorem 3</u>

$$\frac{\partial}{\partial x^i}\left[\sqrt{-g_0}\ \bar{\psi} M^i \psi\right] = 0 \qquad (28)$$

Pf.

$$\frac{\partial}{\partial x^i}\left[\sqrt{-g_0}\ \bar{\psi} M^i \psi\right] = \sqrt{-g_0}\left(\frac{\partial \bar{\psi}}{\partial x^i} M^i \psi + \bar{\psi} M^i \frac{\partial \psi}{\partial x^i}\right)$$

$$+ \bar{\psi} \frac{\partial}{\partial x^i}\left(\sqrt{-g_0}\ M^i\right) \psi$$

$$= 0 \qquad [\text{ by } (22) \ \& \ (26)\ ]$$

§ 5. Eq'n of Neutrino

$m = 0$

i.e. $$\left(M^i \frac{\partial}{\partial x^i} - \frac{2}{R} M^5\right)\psi = 0 \qquad (29)$$

<u>Theorem 4</u>

$$\left(M^i \frac{\partial}{\partial x^i} - \frac{2}{R} M^5\right)\left(M^5 \psi\right) = 0 \qquad (30)$$

$$M^i M^5 \frac{\partial}{\partial x^i}$$

Pf. $$M^i\left(\frac{\partial}{\partial x^i} M^5\right) = \frac{1}{n}\left(\gamma^\mu \frac{\partial x^i}{\partial \xi^\mu}\right)\frac{\partial}{\partial x^i}\left(\gamma^\nu \xi_\nu\right) = \frac{1}{n}\gamma^\mu \gamma^\nu \delta^\alpha_\mu \eta_{\nu\alpha}$$

$$- \frac{1}{n}\gamma^\mu \frac{\partial n}{\partial \xi^\mu}\ \gamma^\nu \frac{\partial \xi_\nu}{\partial n}$$

$$\therefore \quad M^i \frac{\partial}{\partial x^i} M^5 = \frac{1}{R}\gamma^\mu \gamma^\nu \eta_{\mu\nu} - \frac{1}{R} + M^i M^5 \frac{\partial}{\partial x^i}$$

$$M^5 M^i \frac{\partial}{\partial x^i} = \frac{\gamma^\mu \xi_\mu}{R}\left(\gamma^\nu \frac{\partial x^i}{\partial \xi^\nu}\right)\frac{\partial}{\partial x^i}$$

② & $M^5 M^i + M^i M^5 = 0$

$$\therefore \quad M^5 M^i \frac{\partial}{\partial x^i}\psi = -M^i M^5 \frac{\partial}{\partial x^i}\psi = \left[-M^i \frac{\partial}{\partial x^i}(M^5\psi) + \frac{4}{R}\psi\right]$$

$$M^5\left(M^i \frac{\partial}{\partial x^i} - \frac{2}{R}M^5\right)\psi = \left[-M^i \frac{\partial}{\partial x^i} + \frac{2}{R}\right]M^5\psi$$

<u>Therefore</u>, 2-component neutrino is possible !

Actually,

$$M^\mu M^\nu + M^\nu M^\mu = 2\left(\begin{array}{c|c} g^{ij} & \begin{array}{c}0\\0\\0\\0\end{array} \\ \hline 0\;0\;0\;0 & 1\end{array}\right) \qquad (31)$$

§6.

Comparison with Dirac's Method. (Gürsey)

identical with Dirac's result!

<u>Theorem 5</u>. If ψ satisfies (22)

then $$\frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu}\chi = (2 \pm iRm)\chi \qquad (27)$$

where $$\chi = (\psi \pm i M^5 \psi)$$

<u>Pf</u>.

$$M^5 \gamma^\mu\gamma^\nu L_{\mu\nu}\psi = \underbrace{2R\, M^i \frac{\partial\psi}{\partial x^i}}_{\text{by (15)}} = 4M^5\psi + 2Rm\psi$$

i.e. $$\frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu}\psi = 2\psi + Rm\, M^5\psi$$

$$\gamma^\mu\gamma^\nu L_{\mu\nu} M^5\psi = -M^5\gamma^\mu\gamma^\nu L_{\mu\nu}\psi + 8M^5\psi$$
$$= -2Rm\psi + 4M^5\psi$$

i.e. $$\frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu} M^5\psi = -Rm\psi + 2M^5\psi$$

$$\therefore \quad \frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu}\begin{pmatrix}\psi \\ M^5\psi\end{pmatrix} = \begin{pmatrix}2 & Rm \\ -Rm & 2\end{pmatrix}\begin{pmatrix}\psi \\ M^5\psi\end{pmatrix}$$

$$(2-\lambda)^2 + (Rm)^2 = 0$$

or $$\lambda = 2 \pm iRm$$

Eigen fn

$$2a + Rmb = (2 \pm iRm)a$$

i.e. $$\frac{a}{b} = \mp i$$

Remark : if $m=0$ both $\overset{\chi}{\psi}$ & $M^5\overset{\chi}{\psi}$ are eigen-fn.

<u>Direct pf.</u> If $\frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu}\chi = 2\chi$

(then $M^i\frac{\partial}{\partial x^i}\chi = 2M^5\chi$)

therefore, $\frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu} M^5\chi = - M^5\frac{1}{2}\gamma^\mu\gamma^\nu L_{\mu\nu}\chi + 4M^5\chi$

$$= 2M^5\chi$$

#

§7. Comparison with Method of Covariant Derivative.

Def'n Local Coordinates: (a, b, .. i, j vary from 1 to 4)

Choose $e_a^{\ i}$ & $\varepsilon_i^{\ a}$ such that

$$(e_a^{\ i} e_b^{\ j} g_{ij}) = (\eta_{ab}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad (28)$$

$$e_a^{\ i} \varepsilon_i^{\ b} = \delta_a^{\ b} \qquad (29)$$

Therefore,

$$\begin{cases} \varepsilon_i^{\ a} \varepsilon_j^{\ b} g^{ij} = \eta^{ab} \\ g^{ij} = e_a^{\ i} e_b^{\ j} \eta^{ab} \\ g_{ij} = \varepsilon_i^{\ a} \varepsilon_j^{\ b} \eta_{ab} \end{cases} \qquad (30)$$

Def'n. $\alpha^i \equiv e_a^{\ i} \gamma^a \qquad (a = 1, 2, 3, 4) \qquad (31)$

then $\alpha^i \alpha^j + \alpha^j \alpha^i = 2 g^{ij}$

Theorem 6 There exists a matrix S such that

$$\underline{S M^i S^{-1} = \alpha^i} \qquad (32)$$

where $M^i = \frac{\partial x^i}{\partial \xi^\mu} \gamma^\mu \qquad (\mu = 1, 2, \cdots, 5)$

Pf. M^i & α^i obey the same commutation relation.

It is convenient to generalize the 4 (four dim.) unit vectors to 5 dim.

<u>Defn</u> $$e^i_\mu \equiv (e^i_a \ , \ 0) \qquad \text{i.e } e^i_5 = 0 \qquad (33)$$

$$e^5_\mu \equiv (0, 0, 0, 0, 1)$$

$$\alpha^\mu \equiv e^\mu_\nu \gamma^\nu$$

Clearly, $$\alpha^5 = \gamma^5 \quad \& \quad \alpha^i = e^i_a \gamma^a = e^i_\mu \gamma^\mu \qquad (34)$$

$$(\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu) = 2(g^{\mu\nu}) = 2\left(\begin{array}{c|c} g^{ij} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline 0\,0\,0\,0 & 1 \end{array}\right) \qquad (35)$$

Therefore, $$\underline{S M^\mu S^{-1} = \alpha^\mu} \qquad (36)$$

$$S M^5 S^{-1} = \alpha^5$$

<u>Theorem</u> Let $$S\psi \equiv \phi \qquad (37)$$

then $$\underline{\alpha^i \frac{\partial}{\partial x^i}\phi + \alpha^i \Gamma_i \phi = m\phi} \qquad (38)$$

where $$\underline{\alpha^i \Gamma_j - \Gamma_j \alpha^i = \frac{\partial}{\partial x^j}\alpha^i + \Gamma^i_{jk}\alpha^k} \qquad (39)$$

Pf. $$S M^i S^{-1} S \frac{\partial}{\partial x^i}\psi - \frac{2}{R} S M^5 \psi = m\phi$$

or, $$\alpha^i \frac{\partial}{\partial x^i}\phi + \alpha^i \Gamma_i' \phi - \frac{2}{R}\gamma^5 \phi = m\phi \qquad (40)$$

where $$\Gamma_i' \equiv - \frac{\partial S}{\partial x^i} S^{-1} = + S \frac{\partial S^{-1}}{\partial x^i} \qquad (41)$$

$$S M^i S^{-1} = \alpha^i$$

$$S \varepsilon_i^a M^i S^{-1} = \gamma^a \qquad \& \qquad S \gamma^\mu S^{-1} = \alpha^\nu \frac{\partial \xi^\mu}{\partial x^\nu}$$

$$-\Gamma_i' \gamma^a + \gamma^a \Gamma_i' + (S \gamma^\mu S^{-1}) \left(\frac{\partial}{\partial x^i} \varepsilon_j^a \frac{\partial x^j}{\partial \xi^\mu} \right) = 0$$

$$\frac{\partial}{\partial x^i} \left(\varepsilon_j^a \frac{\partial x^j}{\partial \xi^\mu} \right) = -\left(\frac{\partial \varepsilon_j^a}{\partial x^i} \right) \left(\frac{\partial x^j}{\partial \xi^\mu} \right) - \varepsilon_j^a \frac{\partial}{\partial x^i} \left(\frac{\partial x^j}{\partial \xi^\mu} \right)$$

$\therefore$

$$-\Gamma_i' \gamma^a + \gamma^a \Gamma_i' = -\alpha^j \frac{\partial \varepsilon_j^a}{\partial x^i} - \alpha^\nu \varepsilon_j^a \frac{\partial \xi^\mu}{\partial x^\nu} \frac{\partial}{\partial x^i} \left(\frac{\partial x^j}{\partial \xi^\mu} \right)$$

$$e_a^k \frac{\partial \varepsilon_j^a}{\partial x^i} = -\varepsilon_j^a \frac{\partial e_a^k}{\partial x^i} \qquad \therefore \quad \underline{\frac{\partial \varepsilon_j^a}{\partial x^i} = -\varepsilon_k^a \varepsilon_j^b \frac{\partial e_b^k}{\partial x^i}}$$

$$-\Gamma_i' \gamma^a + \gamma^a \Gamma_i' = +\gamma^b \varepsilon_k^a \frac{\partial e_b^k}{\partial x^i} + \alpha^\nu \varepsilon_j^a \Gamma_{i\nu}^j \qquad (42)$$

<u>Defn</u>. $$K_i \equiv -\frac{1}{2} \gamma^5 \gamma_j \varepsilon_a^j \Gamma_{5i}^a \qquad (43)$$

$-\gamma^5 \gamma^a \gamma_j$

$$\therefore \; -K_i \gamma^a + \gamma^a K_i = -\frac{1}{2} \Gamma_{5i}^b \varepsilon_b^j \left(-\gamma^5 \gamma_j \gamma^a + \gamma^a \gamma^5 \gamma_j \right)$$

$$= +\frac{1}{2} \Gamma_{5i}^b \varepsilon_b^a \gamma^5$$

<u>Defn</u>' $$\Gamma_i \equiv \Gamma_i' - K_i \qquad (44)$$

then,

$$\underline{-\Gamma_i \gamma^a + \gamma^a \Gamma_i = \gamma^b \varepsilon_k^a \frac{\partial e_b^k}{\partial x^i} + \alpha^k \varepsilon_j^a \Gamma_{ik}^j} \qquad (45)$$

i.e.

$$\underline{-\Gamma_i \alpha^i + \alpha^j \Gamma_i = \frac{\partial \alpha^j}{\partial x^i} + \Gamma^j_{ik} \alpha^k} \qquad (46)$$

&

$$\alpha^i K_i = -\frac{\alpha^i}{2} \gamma^5 \gamma_j \, \varepsilon_a{}^j \frac{\partial x^a}{\partial \xi^\alpha} \frac{\partial}{\partial x^i} \overset{=\frac{\xi^\alpha}{R}}{\left(\frac{\partial \xi^\alpha}{\partial x^5}\right)}$$

$$= -\frac{1}{2} e^i_b \gamma^b \gamma^5 \gamma_j \, \varepsilon_a{}^j \, \delta^a_{\underline{i}} \, \frac{1}{R}$$

$$= -\frac{1}{2R} \gamma^j \gamma^5 \gamma_j = +\frac{2}{R} \gamma^5$$

$$\therefore \quad \underline{\alpha^i \Gamma_i' = \frac{2}{R} \gamma^5 + \alpha^i \Gamma_i} \qquad (47)$$

Q.E.D.

i.e. Identical with the method of Covariant Derivative.