U To Prof. Pauli

Construction of $\frac{1}{2}$ isospin boson fields

1 — Let us start from a doublet spinor field, say (p, n).
This is described by the 2×2 matrices

$$X = \begin{pmatrix} p_1 & n_1 \\ p_2 & n_2 \end{pmatrix} \quad \text{and} \quad \bar{Z}^+ = \begin{pmatrix} p_3 & n_3 \\ p_4 & n_4 \end{pmatrix} \quad \text{or} \quad Z = \begin{pmatrix} n_4^* & -p_4^* \\ -n_3^* & p_3^* \end{pmatrix}.$$

In a Lorentz transformation, L being unimodular we have

$$X \to LX \quad , \quad Z \to LZ \, .$$

For the baryon transformation

$$X \to e^{i\alpha} X \quad , \quad Z \to e^{-i\alpha} Z$$

For isotope spin transf.

$$X \to XR \quad , \quad Z \to ZR \qquad (R: \text{unitary and unimodular})$$

Under the combined transformations we have

$$X \to L X U \quad , \quad Z \to L Z U^{+}$$

$$\bar{X} \to \bar{U} \bar{X} \bar{L} \quad , \quad \bar{Z} \to U^{+} \bar{Z} \bar{L} \qquad \left(\bar{X} = X^{-1} \operatorname{Det} X = \sigma_2 X^T \sigma_2\right)$$

where $U = R e^{i\alpha}$ so that $UU^{+} = U^{+}U = 1$.

Out of these quantities we can form the boson field

$$B = \bar{X} Z \quad , \quad B \to \bar{R} B R \, .$$

We also have

$$\bar{B} = \bar{Z} X \quad , \quad \bar{B} \to \bar{R} \bar{B} R \, .$$

so that

$\frac{1}{2}(B + \bar{B}) = \frac{1}{2} \operatorname{Tr} B \longrightarrow \frac{1}{2} \operatorname{Tr} B = M_0$ is an isosinglet boson

and $\frac{1}{2}(B - \bar{B}) = B - \frac{1}{2} \operatorname{Tr} B = \vec{\sigma} \cdot \vec{M}$ is an isotriplet boson

We have $B = M^0 + \vec{\sigma} \cdot \vec{M}$ and $\vec{\sigma} \cdot \vec{M} \longrightarrow \bar{R}\, \vec{\sigma} \cdot \vec{M} R$.

2 - Now in order to define a $\frac{1}{2}$ isospin boson field we first define a hermitian 2×2 matrix Φ which is invariant under Lorentz transformations, but which transforms like B under ~~the~~ isospin rotations. It has the properties

$\Phi = \Phi^{+}$, $\Phi\bar{\Phi} = \mathrm{Det}\,\Phi = 0$ and $\Phi \to \bar{R}\,\Phi R$.

We now show the different ways of constructing such a matrix.

a) Such a matrix can be constructed from the fields X and Z.

Let $\vec{M} = \vec{\phi} + i\vec{\chi}$ with ϕ and χ real.

Let $\phi_0^2 = |\vec{\phi}|^2$

Then the matrix $\Phi = \phi_0 + \vec{\sigma}\cdot\vec{\phi}$ has the required properties.

b) Suppose now that X and Z do not represent the proton-neutron doublet, but a lepton doublet with the property that

$$\mathrm{Det}\, Z = 0 .$$

(The electron-neutrino doublet will have this property in the 2-component theory of the neutrino)

To distinguish this from the previous case we denote the doublet by X' and Z'. Hence $\mathrm{Det}\, Z' = 0$.

We have $B' = \bar{X}'Z'$, $\mathrm{Det}\, B' = 0$ $B' \to \bar{R}B'R$.

Let $\Phi' = B'B'^{+}$.

Again Φ' has the same properties as Φ above.

c) We define $B'' = BB'$ where B and B' have been defined above. Then

$$\Phi'' = B''B''^{+} = BB'B'^{+}B^{+} = B\Phi'B^{+}$$

again has the required properties.

We can also choose

$$\Phi''' = B''^{+}B'' = B'^{+}B^{+}BB'$$

3 – Having now constructed a boson field Φ out of one baryon doublet or one lepton doublet or from one baryon and one lepton doublet combined we can associate with Φ a $\frac{1}{2}$ isospin boson field Θ as follows

Because Φ is hermitian, has zero determinant and transforms according to $\Phi \to \bar{R}\,\Phi\, R$ under isorotations, we can always write the matrix Φ in the form

$$\Phi = \Theta\,\Theta^{+}$$

where $\Theta = \begin{pmatrix} \theta^{0}_{*} & 0 \\ \theta^{+} & 0 \end{pmatrix}$ and transforms as $\Theta \to \bar{R}\,\Theta$ under isorotations.

The procedure is exactly the same as in the case of a null vector $A = (a_0, \vec{a})$ so that $a_0^2 - |\vec{a}|^2 = 0$. To A is associated a spinor $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ such that

$$a_0 + a_3 = u_1 u_1^{*} \quad , \quad a_0 - a_3 = u_2 u_2^{*}$$

$$a_1 + i a_2 = u_2 u_1^{*} \quad , \quad a_1 - i a_2 = u_1 u_2^{*} .$$

We have $\dfrac{u_2}{u_1} = \dfrac{a_1 + i a_2}{a_0 + a_3} = \dfrac{a_0 - a_3}{a_1 - i a_2}$; plus the normalization condition for u

Θ is therefore determined up to a phase transformation $\Theta \to \Theta e^{iS}$

We now introduce the unitary unimodular matrix θ defined by

$$\theta = \Theta + \bar{\Theta}^{+} = \begin{pmatrix} \theta^{0} & -\theta^{-} \\ \theta^{+} & \theta^{0*} \end{pmatrix} .$$

We have

$$\Theta = \theta\,\frac{1+\sigma_3}{2} \quad , \quad \bar{\Theta}^{+} = \theta\,\frac{1-\sigma_3}{2} \quad , \quad \Theta \to \bar{R}\Theta \;,\; \bar{\Theta}^{+} \to \bar{R}\bar{\Theta}^{+} .$$

Hence $\Theta^{+} = \dfrac{1+\sigma_3}{2}\,\bar{\theta}$, so that we can write

$$\Phi = \theta\,\frac{1+\sigma_3}{2}\,\bar{\theta}$$ with the transformation law $\theta \to \bar{R}\,\theta$ and the property $\theta = \bar{\theta}^{+}$. We also have

$$\Phi = \theta\,\frac{1+\sigma_3}{2}\,\theta^{+} .$$ The phase transf. is $\theta \to \theta\, e^{i\sigma_3 S}$.

4 - With the help of the field Θ which is a $\frac{1}{2}$ isospin boson we can construct the other strange particles as in Goldhaber's model. Θ has similar properties to Heisenberg's $\tilde{\pi}$.

isotriplet $\quad \vec{\pi} = \vec{\phi}$

isodoublet $\quad K = \Theta \quad$ or $K = \vec{\phi}\,\Theta \qquad S = 1$

strangeness transformation: $\Theta \to \Theta e^{iS}$ or $\theta \to \theta e^{i\sigma_3 S}$.

nucleon isodoublet $\quad \underset{(p,n)}{N} = \begin{cases} X \\ Z \end{cases}$

neutral isosinglet $\quad \Lambda^0 = \begin{cases} X\bar{\Theta}^{+} = X\theta\frac{1-\sigma_3}{2} \\ Z\Theta = Z\theta\frac{1+\sigma_3}{2} \end{cases} \qquad S = -1$

isotriplet $\quad \vec{\Sigma} = \begin{cases} X\vec{\sigma}\bar{\Theta}^{+} = X\vec{\sigma}\theta\frac{1-\sigma_3}{2} \\ Z\vec{\sigma}\Theta = Z\vec{\sigma}\theta\frac{1+\sigma_3}{2} \end{cases} \qquad S = -1$

isodoublet $\quad \Xi = \begin{cases} X\bar{\Theta}^{+}\sigma_1\Theta^{+} = X\theta\frac{1-\sigma_3}{2}\sigma_1\bar{\theta} \\ Z\Theta\sigma_1\bar{\Theta} = Z\theta\frac{1+\sigma_3}{2}\sigma_1\bar{\theta} \end{cases} \qquad S = -2$

The leptons would be described by

isodoublet $\quad (\nu, \varepsilon) = \begin{cases} X' = \begin{pmatrix} \nu_1 & \varepsilon_1 \\ \nu_2 & \varepsilon_2 \end{pmatrix} \\ Z' = \begin{pmatrix} \varepsilon_4^* & -\nu_4^* \\ -\varepsilon_3^* & \nu_3^* \end{pmatrix} \end{cases}$

charged isosinglet $\quad \mu^- = \begin{cases} X'\bar{\Theta}^{+}\sigma_1 = X'\theta\frac{1-\sigma_3}{2}\sigma_1 \\ Z'\Theta\sigma_1 = Z'\theta\frac{1+\sigma_3}{2}\sigma_1 \end{cases} \qquad S = -1$

Unfortunately we also have the possibility of a neutral singlet for a lepton

$\mu^0 ? = \begin{cases} X'\Theta = X'\theta\frac{1+\sigma_3}{2} \\ Z'\bar{\Theta}^{+} = Z'\theta\frac{1-\sigma_3}{2} \end{cases} \qquad S = 1$

and a charged singlet for a baryon. $\Lambda^+ ? \begin{cases} X\Theta\sigma_1 = X\theta\frac{1+\sigma_3}{2}\sigma_1 \\ Z\bar{\Theta}^{+}\sigma_1 = \cancel{X}\theta\frac{1-\sigma_3}{2}\sigma_1 \end{cases} \qquad S = 1$, etc.

5. Reflection properties

As we are using the representation in which γ_5 is diagonal and $\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, under the parity operation $p \to \beta p$ and $n \to \beta n$ we have

parity $$P: \quad X \to \bar{Z}^{+}, \; Z \to \bar{X}^{+}$$

Under charge conjugation $p \to \gamma_2 p^*$, $n \to \gamma_2 n^*$ we have

$$C: \quad X \to Z\sigma_2 \;, \quad Z \to X\sigma_2 \,.$$

Now the G conjugation introduced by Yang & Lee is the product of C with a isorotation of $\frac{\pi}{2}$ round the Oy axis. Let this special isorotation be represented by $R_{(2)}$. We have

$$R_{(2)}: \quad X \to X e^{i\sigma_2 \frac{\pi}{2}} = X i\sigma_2 \;, \quad Z \to Z i\sigma_2 \,.$$

Hence the G conjugation is represented by

$$G: \quad X \to iZ \;, \quad Z \to iX$$

Now the special baryon gauge transformation with the ~~phase~~ $\frac{\pi}{2}$ is represented by

$$B: \quad X \to e^{i\frac{\pi}{2}} X = iX \;, \quad Z \to -iZ \,,$$

so that we can define another reflection operation G' by

modified charge conjugation $$G' = GB: \quad X \to +Z \;, \quad Z \to -X \,.$$

We also note the combined operations:

$$PG: \quad X \to i\bar{X}^{+} \;, \quad Z \to i\bar{Z}^{+}$$

and $$PG': \quad X \to \bar{X}^{+} \;, \quad Z \to -\bar{Z}^{+} \,.$$

Now the nucleon meson interaction is invariant under both G' and PG'. The weak interactions which only contain X are only PG' invariant (which means PC invariance because of isotopic spin invariance and baryon conservation).

On the other hand G' and PG' invariance combined lead to the separate conservation of P and C which seems to be a general property of the strong interaction. Therefore the question arises whether there is any link between G' conservation and strangeness conservation. To answer this question we are led to study the reflection properties of the $\frac{1}{2}$ isospin boson field Θ which produces strangeness.

We have seen that Θ is defined by means of the field

$$\vec{\phi} = \bar{X} Z - \bar{Z} X + X^{+} \bar{Z}^{+} - Z^{+} \bar{X}^{+} .$$

Under parity:

$$P: \quad \vec{\phi} \to - \vec{\phi} \qquad\qquad i\vec{\phi} \to i\vec{\phi}$$

Under the modified charge conjugation

$$G': \quad \vec{\phi} \to \vec{\phi}$$

Now we construct the matrix

$$\Phi = |\phi| + \vec{\sigma} \cdot \vec{\phi}$$

which is invariant both under P and G'.

We have

$$\Phi = \Theta \Theta^{+} = \theta \frac{1+\sigma_3}{2} \theta^{+} .$$

$$\text{Under } G: \quad \Theta \to \Theta^{+} , \quad \theta \to \theta^{+} = \bar{\theta} .$$

$$\text{Under } P: \quad \Theta \to \Theta$$