Rest Frame Rotations of spin $\frac{1}{2}$ Particles
(Spatial Interpretation of Some Canonical Transformations)

1. Introduction.

The extension of classes of possible interactions brought about by non conservation of parity and charge conjugation in weak interactions has revived interest in the study of canonical transformations involving charge conjugate and parity conjugate ($\gamma_5 \psi$) functions. The purpose of this note is to show the existence of a group of transformations isomorphic to the Lorentz group for the wave function of a fermion, this group involving gauge transformations $i\gamma_5$ transformations and Pauli's $i\gamma_5 \Gamma$ (Γ = charge conjugation) as subgroups. The zero mass Dirac equation is shown to be invariant with respect to these transformations. For a fermion with mass the term involving the mass is changed under the group. It is shown that physically the new group may be interpreted as a Lorentz transformation in the rest frame of the fermion ~~where the γ_4 - current vector reduces to its~~

We use ^covariant^ γ operators such that $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}$
and $\gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \quad (\gamma_5^2 = 1)$

2 – ~~The Pauli group and~~ The phase group and the Pauli group ~~and the A group~~

Let a 4-spinor wave function ψ undergo the transformation

$$\psi \to S\psi \qquad (1)$$

The particular transformation

$$S_0 = e^{i\gamma_5 a} = \cos a + i\gamma_5 \sin a \quad (a\ \text{real}) \qquad (2)$$

has been used by the author ( ), Touschek (2), Nishijima( ) and others. Pauli ( ) has recently introduced the transformation

$$S'\psi = a\psi + b\gamma_5 \psi^c \qquad (3)$$

where a and b are complex and satisfy the condition

$$|a|^2 + |b|^2 = 1 .$$

and $\psi^c = C\psi^*$ is the charge conjugate function, C being a matrix such that

$$\gamma_\mu^T = - C \gamma_\mu C^{-1}.$$

~~In particular $C = \gamma_2$.~~ The charge conjugation operator may be written as

$$\Gamma = CK \qquad (4)$$

where K is the operation of complex conjugation.

We now introduce the operators

~~$T_3 = \gamma_5$~~ ~~5a~~

$$T_1 = i\Gamma = iCK \qquad 5a$$

$$T_2 = \Gamma = CK \qquad 5b$$

~~and~~ $$T_3 = \gamma_5 \qquad 5c$$

and $$j = i\gamma_5 \qquad 5d.$$

We have: $$T_1^2 = T_2^2 = T_3^2 = 1 \qquad 6a$$

$$T_1 T_2 = - T_2 T_1 = j T_3 \quad , \text{etc.} \qquad 6b$$

~~and~~

The corresponding 4-spinor ψ is given by

$$\psi_1 = \xi_1 \ , \ \psi_2 = \xi_2 , \ \psi_3 = \eta_1 \ , \ \psi_4 = \eta_2 .$$

We now define a 2×2 matrix Ψ by

$$\Psi = \begin{pmatrix} \xi_1 & \eta^1 \\ \xi_2 & \eta^2 \end{pmatrix} = \begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix} = \begin{pmatrix} \psi_1 & -\psi_4^* \\ \psi_2 & \psi_3^* \end{pmatrix}.$$

~~The corresponding~~

We choose for the Dirac matrices $\alpha_n (= \gamma_0\gamma_n)$ and $\beta (= \gamma_0)$ the Weyl representation.
Since $\gamma_5 = -i\alpha_1\alpha_2\alpha_3$ we have the matrices

$$\alpha_n = \begin{pmatrix} \sigma_n & 0 \\ 0 & -\sigma_n \end{pmatrix} , \quad \beta = \begin{pmatrix} 0 & \underline{1} \\ \underline{1} & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{pmatrix}$$

where $\underline{1}$ is the 2×2 unit matrix and the σ's denote the Pauli matrices.
~~We have $\gamma_5 = -i\alpha_1\alpha_2\alpha_3 = \begin{pmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{pmatrix}$~~ We now choose $C = \alpha_2\beta = -\gamma_2$ for the charge conjugation matrix.
It can be shown easily that we have the following correspondence *

$$\alpha_n \psi = \gamma_0\gamma_n \psi \longleftrightarrow \sigma_n \Psi$$

$$\beta\psi = \gamma_0\psi \longleftrightarrow \sigma_2 \Psi^* \sigma_2 = (\bar{\Psi})^\dagger$$

$$\psi^* \longleftrightarrow \Psi^* \sigma_2$$

$$T_n \psi \longleftrightarrow \Psi \sigma_n$$

$$i\gamma_5\psi = j\psi \longleftrightarrow i\Psi$$

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* If the usual representation is used for α_n and β, the ~~above~~ correspondence ( ) still hold provided the relation between Ψ and ψ is given by ~~Ψ is replaced by~~

$$\Psi = \begin{pmatrix} \psi_1 + \psi_3 & -\psi_2^* + \psi_4^* \\ \psi_2 + \psi_4 & \psi_1^* - \psi_3^* \end{pmatrix}$$

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where $\bar{\Psi}$ means the adjoint matrix (explicitly $\bar{\Psi} = \Psi^{-1}\,\mathrm{Det}(\Psi)$), the dagger ~~and ... denotes respectively~~ Hermitian conjugation. It follows that
The Dirac equation in 2×2 matrix notation can be written as

$D\Psi = m$
where
$D = \partial_0 + \vec{\sigma}\cdot\vec{\nabla}$
$A = A_0 + \vec{\sigma}\cdot\vec{A}$

$$(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\Psi = \left\{ m(\bar{\Psi})^\dagger + e\,(A_0 + \vec{\sigma}\cdot\vec{A})\Psi \right\} i\sigma_3 \qquad ( )$$

The properties of this representation and the above correspondences have been fully discussed elsewhere. ( ).

and $\qquad j^2 = -1 \qquad j = T_1 T_2 T_3$ $\qquad$ 6c.

the operator j commuting with T_1, T_2 and T_3. It is clear that T_1, T_2 ~~and~~, T_3 and j are isomorphic to the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ and the imaginary unit i respectively. Hence, corresponding to the unitary 2×2 matrix

$$U^{u} = e^{i\lambda} e^{i\vec{\mu}\cdot\vec{\sigma}} \qquad (7)$$

~~we have~~ the where λ is a real scalar and $\vec{\mu}$ is a real vector we have the corresponding unitary operator

$$U' = e^{j\lambda} e^{j\vec{\mu}\cdot\vec{T}} \qquad (8)$$

It can be verified immediately that the transformation

$\psi \to U'\psi$ is the same as

$$\psi \to S_0 S' \psi \qquad (9)$$

which is the product of the $i\gamma_5$ transformation (2) and Pauli's transformations (3). In this formalism it is obvious that the transformations (9) form a ~~group~~ 4 parameter group isomorphic to the group of 2×2 unitary matrices. Hence, Pauli's group (3) is isomorphic to three dimensional rotations and the $i\gamma_5$ transformation is isomorphic to phase transformations. ~~or gauge transf~~

From the foregoing ~~our~~ analysis it follows that it is possible to devise a formalism in which the group introduced by Pauli is represented by the matrices σ_m and the imaginary unit i. ~~We proceed as follows~~ This will be done in the next section

3 - The 2×2 matrix representation.

Weyl ( ) has shown that the Dirac equation can be written by means of the 2-spinors ξ_i and η_i $(i = 1, 2)$. P.T.O.

If we now define a 2×2 matrix Ψ by

$$\Psi = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \qquad (10)$$

then, using the relations

$$\xi_1 = \psi_1 + \psi_3 \qquad \eta_1 = -\psi_2^{\times} + \psi_4^{\times} \qquad (11)$$
$$\xi_2 = \psi_2 + \psi_4 \qquad \eta_2 = \psi_1^{\times} - \psi_3^{\times}$$

which hold between the Dirac 4-spinor ψ and the Weyl 2-spinors ξ and η (See for instance . .) it can be shown easily that we have the following correspondences

$$\alpha_n \psi = \gamma_0 \gamma_n \psi \longleftrightarrow \sigma_n \Psi \qquad (12a)$$
$$\beta \psi = \gamma_0 \psi \longleftrightarrow \sigma_2 \Psi^{*} \sigma_2 = (\bar{\Psi})^{\dagger} \qquad (12b)$$
$$T_n \psi \longleftrightarrow \Psi \sigma_n \qquad (12c)$$
$$j \psi \longleftrightarrow i \Psi. \qquad (12d)$$

where $\bar{\Psi}$ means the adjoint matrix (defined by $\bar{\Psi} = \Psi^{-1} \det(\Psi)$) the dagger and T denotes respectively hermitian conjugation and transposition. These correspondences have already been established and used elsewhere ( ).

Under a Lorentz transformation L, the transformation law for ψ is

$$L: \qquad \psi' = L\psi = e^{\frac{1}{2} a_{\mu\nu}\sigma_{\mu\nu}} \psi \qquad (13)$$

where $\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$ and the parameters $a_{\mu\nu}$ are real. This can also be written as

$$\psi' = e^{\frac{i}{2}\vec{a}\cdot\vec{\sigma}}\, e^{\vec{b}\cdot\beta\vec{\sigma}} \psi \qquad (13a)$$

The components of $\vec{\sigma}$ are the Pauli matrices doubled and $\vec{b}$ represent the direction and magnitude of spatial rotation while $\vec{a}$ determines a pure Lorentz transformation.

§4 - The groups Λ, Λ' and G.

We now consider the following six parameter group of transformation Λ which we call Λ transformation (or internal Lorentz transformations) in contrast with the ordinary (or external) Lorentz transformations.

$$\Lambda : \quad \psi'' = \Lambda\psi = e^{\vec{\lambda}\cdot\vec{T}}\, e^{\vec{\mu}\cdot j\vec{T}}\,\psi . \qquad (14)$$

In the 2×2 matrix representation we have

$$L : \Psi \rightarrow M\Psi \qquad (15)$$

$$\Lambda : \Psi \rightarrow \Psi N \qquad (16)$$

where M and N are unimodular matrices such that $\mathrm{Det}\,M = \mathrm{Det}\,N = 1$, while the representation of the phase transformation (2) is

$$\Pi : \psi \rightarrow e^{ja}\psi \qquad (17)$$

or $$\Pi : \Psi \rightarrow e^{ia}\Psi \qquad (17a)$$

in the 4-spinor and the 2×2 matrix representations respectively. The transformations Λ and Π form the 7 parameter group Λ'. Combining (13), (14) and (17) ~~we obtain a 13 parameters group~~ because of the commutation of L, Λ and Π we obtain a 13 parameter group, namely

$$G : \quad \Psi \rightarrow e^{ia} M \Psi N \qquad (18)$$

Under this group, the quantity

$$|q|^2 = |\mathrm{Det}\,\Psi|^2 = (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 \qquad (19)$$

is an invariant.

If we restrict ourselves to the group $L\Lambda$, the complex number

$$q = \bar{\psi}\psi + i\,\bar{\psi}\gamma_5\psi \qquad (20)$$

is invariant so that $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$ are invariant separately under $L\Lambda$.

A subgroup ~~of~~ of the seven parameter group Λ is the group of unitary 2×2 matrices U with the property

$$UU^{\dagger} = 1 . \qquad (21)$$

Therefore it is isomorphic to the product of phase transformations and spatial rotations. Its 2×2 representation is ~~$U =$~~

$$U = e^{ia} e^{i\vec{k}\cdot\vec{\sigma}} . \qquad (22)$$

The general group G may therefore be written as

$$G: \quad \Psi \to L \Psi U H$$

where U is unitary and H is a hermitian unimodular 2×2 matrix. The combination of the Lorentz transformation with the Pauli group will be denoted by

$$G': \quad \Psi \to L \Psi U \qquad (23).$$

We note that the gauge transformation

$$\psi \to e^{is} \psi \qquad (24)$$

~~is represented by~~ can be written as

$$\psi \to e^{i\tau_3 s} \psi \qquad (24a)$$

so that the matrix N which multiplies Ψ on the right becomes in this case

$$N = e^{i\sigma_3 s} . \qquad (24b)$$

This is a special unitary matrix of the type (22) ~~and~~ which shows that the gauge transformation belongs to the Pauli group. So does the $i\gamma_5$ transformation. ~~But~~ The charge conjugation which can be written as

$$\psi \to \psi^c = \cancel{i e^{i\tau_2 \frac{\pi}{2}}} \psi = \tau_2 \psi = e^{i\pi \frac{1+\tau_2}{2}} \psi$$

$\cos iy = \cosh y$

$\sin iy = i \sinh y$

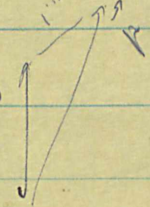
$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$(\xi - i\xi^3) \times i\lambda_2 (c_2 - \xi c_3)\,)(\xi^* + \xi\xi^3 \ldots$

$(c_1 + i\lambda c_2)(\bar{c}_1 + \bar{c}_2 i\lambda_2) = c_1\bar{c}_1 + c_2\bar{c}_2 + i\lambda_2(\bar{c}_2 c_1 + \bar{c}_1 c_2)$

$c_1 \bar{c}_2 c_2 = i c_2 \bar{c}_1 c_2 \quad + 2 i\lambda \bar{c}_1 c_2$

$i\lambda_2(\bar{c}_1 c_2 + \bar{c}_2 c_1 + \bar{c}_1 c_2 - \bar{c}_2 \bar{c}_1)$



$-\Lambda e_3 \bar{\Lambda}$

$i\xi_3\, \xi^{-i\xi_3}$

$\Lambda^{+} \varphi e_3 \bar{\Lambda}$

$-i\xi_3 \quad e_3 \quad \bar{\Lambda} i\xi_3$

$D^{+} = m\psi^{+}$

$\psi\Lambda$

$\bar{\psi} = \bar{\Lambda}\psi$

$\bar{\psi}^{+} = \bar{\psi}^{+}\bar{\Lambda}^{+}$

is also a special case of (22) since the corresponding N matrix is

$$N = e^{i\pi \frac{1+\sigma_2}{2}} = \sigma_2, \qquad (25).$$

But the transformation

$$\psi \to \psi \cosh\alpha + \psi^c \sinh\alpha$$

is represented by the hermitian unimodular matrix

$$\to N = e^{\sigma_2 \alpha}$$

and belongs to the extended group Λ.

5 – Transformation properties of the Dirac equation under the group Λ.

The transformation properties of the Dirac equation are best discussed in the 2×2 matrix form (). ~~Using the correspondences ( ) Dirac's equation may be written as~~

$$\cancel{D\Psi = m\bar{\Psi}^{+} i\sigma_3}$$

Then if we write $\Psi' = \Psi\Lambda$, we obtain

$$D\Psi' = m\bar{\Psi}'^{+}\bar{\Lambda}^{+} i\sigma_3 \bar{\Lambda}^{-1} + eA\Psi'\Lambda i\sigma_3 \bar{\Lambda}^{-1}.$$

In particular if $\Lambda = \exp(i\sigma_3 S)$ (gauge transformation) we have the same equation as before. If $\Lambda = \sigma_2$, the equation obeyed by ψ' (which now means the charge conjugate) is still the same for the free particle, but the electromagnetic term changes its sign. If $\Lambda = \sigma_3$ then Ψ' corresponds to the spinor $\gamma_5\psi$, which satisfies the same equation with the rest mass changed.

~~In the case of the neutrino, however~~

The phase transformation in which $\psi \to e^{i\gamma_5 \lambda}\psi$ we have $\Lambda = \exp i\lambda$ and again the only effect of this transformation is the multiplication of the rest mass term by $e^{2i\lambda}$ ($\exp 2i\gamma_5\lambda$ in the 4-spinor equation). Then, except for the gauge transformations Ψ' satisfies a different equation. In the case of the neutrino, however, $e = m = 0$ and ψ' satisfies the same equation. Therefore the neutrino equation is invariant under the 13 parameter transformation

$$\Psi \to L \Psi \Lambda e^{i\lambda}.$$

We now turn to the transformation properties of the bilinear quantities. It has been shown elsewhere that the 2×2 matrices

$$U^{(0)} = \Psi\Psi^+ , \quad U^{(n)} = \Psi \sigma_n \Psi^+$$

represent 4 mutual orthogonal 4-vectors with components

$$U^{(0)}_\mu = \bar\psi \gamma_\mu \psi, \quad U^{(3)}_\mu = \bar\psi \gamma_5 \gamma_\mu \psi, \quad U^{(1)}_\mu + iU^{(2)}_\mu = \bar\psi \gamma_\mu \psi^c .$$

so that $U^{(0)}$ and $U^{(3)}$ are essentially the current and spin 4-vectors respectively. $U^{(1)}$ and $U^{(2)}$ are the de Broglie vectors. Under Lorentz transformations we see that

$$U^{(\nu)} \to L U^{(\nu)} L^+ \qquad (\nu = 0, 1, 2, 3)$$

Let us now study the effect of Λ transformations. In a gauge transformation $U^{(0)}_\mu \to U^{(0)}_\mu$ and $U^{(3)}_\mu \to U^{(3)}_\mu$

$$U^{(0)}_\mu \to (\cos S)\, U^{(0)}_\mu + \sin(S)\, U^{(3)}_\mu$$

$$U^{(3)}_\mu \to -\sin S\, U^{(0)}_\mu + \cos S\, U^{(3)}_\mu .$$

the gauge invariant vectors $U^{(0)}$ and $U^{(3)}$ are not changed but $U^{(1)}$ and $U^{(2)}$ are transformed into each other according to the law

In the case of the Lee-Yang neutrino $\psi_\nu = \gamma_5 \psi_\nu$ so that $\Psi_\nu = \Psi \frac{1+\sigma_3}{2}$. ~~For the case of~~ The equation is $D\Psi\left(\frac{1+\sigma_3}{2}\right) = 0$. (In the case of the Majorana neutrino we have $D\Psi\frac{1+\sigma_2}{2}$. The two are related by a Λ transformation namely $\Psi \to \Psi e^{i\sigma_1 \frac{\pi}{2}}$

$$U^{(1)} + iU^{(2)} \to e^{2iS}\left(U^{(1)} + iU^{(2)}\right).$$

~~For a unitary transformation of the Pauli type~~

As for a general Λ transformation

$$\Psi\Psi^{+} \to \Psi\Lambda\Lambda^{+}\Psi^{+}$$

$$\Psi\sigma_n\Psi^{+} \to \Psi\Lambda\sigma_n\Lambda^{+}\Psi^{+}.$$

For a Pauli transformation $U^{(0)}$ is not changed, but $U^{(1)}, U^{(2)}, U^{(3)}$ are rotated into each other in a way that preserve $U_\mu^{(1)2} + U_\mu^{(2)2} + U_\mu^{(3)2}$.

If Λ is of the form $\Lambda = e^{\vec{\sigma}\cdot\vec{a}}$, then all the ~~four~~ 4-vectors transform in a way which is similar to a Lorentz transformation in the bracketted indices. Phase $(i\sigma_3)$ transformations leave all the 4-vectors invariant.

Finally if Λ is hermitian we have transformations similar to Lorentz transformations which ~~[illegible]~~ affect $U^{(0)}$ as well as $U^{(n)}$. For instance for $\Lambda = \exp\sigma_3\omega$ we have

$$U^{(0)} \to \cosh\omega\, U^{(0)} + \sinh\omega\, U^{(3)}$$

$$U^{(3)} \to \sinh\omega\, U^{(0)} + \cosh\omega\, U^{(3)}.$$

If $\Lambda = \sigma_2$ (charge conjugation)

$$U^{(0)} \to U^{(0)},\quad U^{(3)} \to -U^{(3)},\quad U^{(1)} \to -U^{(1)},\quad U^{(2)} \to U^{(2)}$$

Similarly we can form the complex invariant

$$\bar{\Psi}\Psi = \bar{\psi}\psi + i\bar{\psi}\gamma_5\psi \qquad (\;)$$

and the ~~[illegible]~~ matrices

$$\bar{\Psi} i\sigma_3 \Psi = \sigma_\mu\bar{\sigma}_\nu\left(\bar{\psi}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi\right) = \sigma_\mu\bar{\sigma}_\nu M_{\mu\nu} = \sigma_\mu\bar{\sigma}_\nu M^{(3)}_{\mu\nu}$$

$$\bar{\Psi} i\sigma_1 \Psi = \sigma_\mu\bar{\sigma}_\nu \,\mathrm{Re}\,\bar{\psi}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi^c = \sigma_\mu\bar{\sigma}_\nu M^{(1)}_{\mu\nu}$$

$$\bar{\Psi} i\sigma_2 \Psi = \sigma_\mu\bar{\sigma}_\nu \,\mathrm{Im}\,\bar{\psi}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi^c = \sigma_\mu\sigma_\nu M^{(2)}_{\mu\nu}$$

Λ transformations do not affect the invariant ( ) while under the phase transformation we have

$$\bar\psi\psi \rightarrow \cos\lambda\, \bar\psi\psi + \sin\lambda\, \bar\psi\gamma_5\psi$$

$$\bar\psi\gamma_5\psi \rightarrow -\sin\lambda\, \bar\psi\psi + \cos\lambda\, \bar\psi\gamma_5\psi .$$

Under Λ transformation we get

$$\bar\psi\, i\sigma_m \psi \rightarrow \bar\psi\bar\Lambda\, i\sigma_m \Lambda\psi$$

while under a phase transformation ~~undergoes~~ the antisymmetrical ~~$M_{\mu\nu}$~~ tensors $M^{(m)}_{\mu\nu}$ undergo the phase transformations (called dual rotation in a previous paper)

$$M^{(m)}_{\mu\nu} \rightarrow \cos 2\lambda\, M^{(m)}_{\mu\nu} + \sin 2\lambda\, \tilde M^{(m)}_{\mu\nu} .$$

These transformations are remarkable in ~~that~~ one respect. They leave invariant the Maxwell tensors which one may construct from the corresponding antisymmetrical tensors. Incidentally we have shown that the $i\gamma_5$ transformation ~~6 - Interpretation of the~~ of spinors induces a dual rotation of antisymmetrical tensors.

6 - Interpretation of the Λ transformations as rest frame Lorentz transformations.

The solution of the Dirac equation in the field free case reads in 2×2 matrix notation

$$\Psi = \Phi e^{i\sigma_3 S} K$$

where $S = p_\nu x_\nu = \frac{1}{2}Tr(\bar P X)$, $\bar P = \bar\Phi^{+}\Phi^{-1}$

and K such that $\bar K^{+}\sigma_3 = \sigma_3 K$.

For a given energy momentum P ~~there~~ we find

$$\Phi = \left(\frac{P}{m}\right)^{\frac{1}{2}} R$$

where R is a unimodular unitary matrix and the general expression for K is

$$K = c_3 + i\sigma_3 c_3' + \sigma_2 (c_4 + i\sigma_4 c_4').$$

Putting

$$C_3 = c_3 + i c_3' \quad \text{and} \quad C_4 = c_4 + i c_4'$$

we get for the 4 spinor form of (?)

$$\psi = C_3 \varphi e^{iS} + C_4 \varphi^c e^{-iS}$$

so that C_3 and C_4 are the amplitudes for positive and negative energies respectively.

To determine the wave function further we would give the sign of the energy and the direction of polarization. For instance a positive energy electron with polarization along $\vec{\xi}$ in the frame of reference in which $\vec{p} = 0$ is given by $\Phi \sigma_3 \Phi^+ = R \sigma_3 \bar{R} = \vec{\xi} \cdot \vec{\sigma}$.

and the solution is

$$R(\xi) = \frac{1 ~~+~~ - (\vec{\xi} \cdot \vec{\sigma}) i\sigma_3}{(\quad)}.$$

And as we have

$$P^{\frac{1}{2}} = \frac{m + E + \vec{\sigma} \cdot \vec{p}}{\sqrt{\quad}}$$

we have

$$\Psi_+(\vec{p}, \vec{\xi}) = P^{\frac{1}{2}} R(\vec{\xi}) e^{i\sigma_3 P \cdot X}.$$

We now define the rest system of the electron as the frame of reference in which its ~~total~~ ~~the~~ ~~velocity~~ the expectation value of its velocity vanishes.

as well as its momentum $\vec{p}$.

If $\vec{p} = 0$ we have

$$U^{(0)} = \psi\psi^{+} = R(\vec{\xi})\, e^{i\sigma_3 S} K K^{+} e^{-i\sigma_3 S} \bar{R}(\vec{\xi})$$

$$K K^{+} = |c_3|^2 + |c_4|^2 + 2\sigma_2 \,\mathrm{Re}\, c_3^{*} c_4 + 2\sigma_1 \,\mathrm{Im}\, c_3^{*} c_4$$

so that in the rest frame we must have

$$c_1^{*} c_2 = 0 \qquad (\ )$$

That is the electron is either in a positive or negative energy state. If condition ( ) is not satisfied the the electron with $\vec{p} = 0$ still possesses a Zitterbewegung. We now rotate the axes as to make the Oz axis coincide with the polarization vector $\vec{\xi}$. Then $R = 1$.

In this frame, if the

$$\psi_{+} = e^{i\sigma_3 S} c_1 \quad \text{or} \quad \psi_{-} = e^{i\sigma_3 S} \sigma_2 c_2$$

according as the energy is positive or negative. Then the vectors $U^{(0)}, U^{(n)}$ are given by

$$U^{(0)} = 1$$

$$U^{(1)} + iU^{(2)} = e^{2i\sigma_3 S} \sigma_1 = e^{2i\sigma_3 m s} \sigma_1$$

$$U^{(3)} = \sigma_3$$

We see that in this frame $U^{(0)}$ and $U^{(3)}$ correspond respectively with the time and Oz axes and that $U^{(1)}$ and $U^{(2)}$ rotate round the Oz axis with angular velocity $2m$. Now, ~~the~~ rotation in this frame a rotation round the Oz axis is equivalent to a gauge transformation. ~~Thus~~ In the frame of reference rotating with angular velocity $2m$ the spinor has the form $\psi_{+} = c_1$ or $\psi_{-} = \sigma_2 c_2$.

that is, the positive energy spin has a component longitudinal with respect to its polarization vector and the negative energy spin is transverse. In this frame now all the four vectors $U^{(0)}$ and $U^{(n)}$ are at rest.

The physical meaning of this rest system is follows. Using the general solution ψ we find that the velocity 4-vector $U^{(0)}$ is not constant because of the Zitterbewegung. Now this zitterbewegung causes the electron to have an acceleration. We now want to define an operator B such that (acceleration 4-vector)

$$\langle \bar{\psi} B_\mu \psi \rangle = \frac{E}{m}\frac{d}{dt} \langle \bar{\psi} \gamma_\mu \psi \rangle$$

We find, differentiating the expression ( ) for $U^{(0)}$

$$\frac{E}{m}\frac{dU^{(0)}}{dt} = 2\sigma_3 \left(2\sigma_2 \, \mathrm{Re}\, c_3^* c_4 + 2\sigma_1 \, \mathrm{Im}\, c_3^* c_4 \right)$$
$$= 4 \left(\sigma_2 \, \mathrm{Re}\, i c_3^* c_4 + \sigma_1 \, \mathrm{Im}\, i c_3^* c_4 \right)$$

This is always orthogonal to both the velocity and the spin 4-vectors. We ~~find~~ can write

$$\langle \bar{\psi} B_\mu \psi \rangle = \mathrm{Re}\, e^{i\alpha} \langle \bar{\psi} \gamma_\mu \psi^c \rangle$$

We can now choose the gauge α so that

$$\mathrm{Re} \langle \bar{\psi} B_\mu \psi \rangle = \mathrm{Re} (\bar{\psi} \gamma_\mu \psi^c)$$

In 2×2 matrix notation the acceleration vector is the

$$B = \Psi i e_1 \Psi^{+}.$$

To sum up we can always find a frame for the free electron in which the expectation value of the velocity vanishes, the expectation value of the acceleration due to Zitterbewegung points along the Ox axis and the spin expectation value is in the Oz direction. In this frame

The spinor is $\Psi_{\text{rest}} = 1$ ~~or $\Psi_- = \sigma_2$.~~

~~The orthogonal frame of reference associated with Ψ_+ is right handed ($1, \sigma_1, \sigma_2, \sigma_3$) whereas the orthogonal frame one associated with Ψ_- is ($1, -\sigma_1, \sigma_2, -\sigma_3$)~~

~~If now we apply a Λ transformation.~~

Let

$$\Psi(t) = L(t)\,\Psi_{\text{rest}}.$$

L, being the Lorentz transformation which transforms Ψ_{rest} into a solution of the Dirac equation with definite directions for the momentum, the polarization and the acceleration.

Now if we perform a Λ operation on $\Psi(t)$ we have

$$\Psi \to \Psi(t)\Lambda = L(t)\Lambda\Psi_{\text{rest}}$$

so that we can write

$$\Psi\Lambda = L(t)\,\Lambda\,L(t)^{-1}\,\Psi(t)\,!$$

This shows that a Λ transformation is equivalent to a Lorentz transformation applied to the wave function of the electron in its rest system (where the expectation values of the velocity is zero and the expectation values of, the Zitter-acceleration and the spin are respectively along specified orthogonal space axis.

In particular the gauge transformation is a rotation round the spin axis in this system and the charge conjugation is a rotation of π ~~round~~ in the plane ~~of~~ defined by the spin and the acceleration vectors followed by a phase transformation which has no effect on the vectorial bilinear quantities.

Of Course the electron can never be brought to rest in the above sense, as the acceleration is not observable, but is a consequence of the Zitterbewegung which ~~is caused by~~ has its origin in the uncertainty relation. But this concept provides a way of interpreting and classifying various Lor. invariant transformations in terms of space-time properties. For instance if a quantity is Λ invariant then it means that in the rest frame of the electron it has maximum space-time symmetry. If it is invariant with respect to the Pauli transformation only, then it lacks 4-dimensional symmetry but it is isotropic in three dimensional space.

7- ~~Sy~~ Invariant ~~Symmetry~~ properties.

Now the space-time symmetry properties of physical quantities may be described by means of their behavior under Λ transformations.

In analogy with Lorentz transformation we may define ~~a Λ-Spinor which~~ Lorentz invariant 2×2 matrices which represent, a Λ-~~Scalar~~, etc. by the following transformation properties

Λ-Spin: $\Xi \to \Xi\Lambda$

Λ-scalar: $\Omega \to \Omega$.

Λ-tensor $Q \to \bar{\Lambda} Q \bar{\Lambda}$

Λ-vector $U \to \Lambda U \Lambda^{+}$.

where Ξ, Ω, Q, U are 2×2 matrices Q being traceless

For instance the ~~sp~~ Dirac spinor Ψ which transforms as

$$\Psi \rightarrow L \Psi \Lambda$$

behaves as a spinor both under Lorentz and Λ transformations.

Given 2 Dirac spinors Ψ and Φ we can construct the following invariants

$$\Omega(\Psi \bar{\Phi} + \Phi \bar{\Psi}) = \Omega \tfrac{1}{2} \mathrm{Tr}(\bar{\Psi} \Phi)$$

where Ω is a Λ scalar. We can construct more Lorentz and Λ invariants if we dispose of A-vectors and Λ-vectors, namely

$$\mathrm{Re}\,\mathrm{Tr}(\Psi Q \bar{\Phi}) \quad \text{and} \quad \mathrm{Re}\,\mathrm{Tr}(\Psi U \bar{\Phi})$$

~~8. Application to Fermi interactions and Yukawa interaction~~

~~8. The neutrino.~~

8. Application to Fermi and Yukawa interaction.

We now try to construct an interaction term which is real Lorentz invariant and Λ invariant, coupling the fields S, P, T, A, V (respectively scalar, pseudoscalar, tensor, axial vector and vector) to the spinor fields Ψ and Φ. ~~We represent the scalar field S by the m~~ It is convenient to represent the fields S, P and T as a single matrix F defined as.

$$F = S + iP + \tfrac{i}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \sigma_\mu) T^{\mu\nu} = S + iP + T.$$

so that we have $\frac{1}{2} \mathrm{Tr}\, F = S + iP$.

We also group A and V into the single matrix $E = (A_\mu + iV_\mu)\sigma^\mu$

so that

$$\mathcal{H}\, E = A_\mu \sigma_\mu \quad \text{and} \quad \mathcal{H}\, iE = -V_\mu \sigma_\mu$$

where $\mathcal{H}$ denotes the Hermitian part of the matrix which follows.

~~The~~ The behaviour of F and E under the Lorentz transformation is given by

$$F \to L F \bar{L} \quad \text{and} \quad E \to L E L^{+}.$$

We now choose ~~a~~ Lorentz invariant matrices G_{STP} and G_{AV}.

Then the following matrices

$$\Psi G_{STP} \bar{\Phi} \quad \cancel{\text{and}}, \quad \bar{\Psi}^{+} G_{VA} \bar{\Phi}, \quad \Phi G'_{STP} \bar{\Psi} \text{ and } \bar{\Phi}^{+} G'_{VA} \bar{\Psi}$$

have same transformation properties as F and $\bar{E}^{+}$.

Here G'_{STP} transforms like $\bar{G}_{STP}$ and G'_{VA} like G^{+}_{VA}.

Hence

$$L_{int} = \mathcal{R}\, Tr\{F \Psi G_{STP} \bar{\Phi} + E \bar{\Psi}^{+} G_{VA} \bar{\Phi}\}$$

is a real Lorentz invariant quantity.

For the special case

$$G_{STP} = g_S + i g_P + i\sigma_3 g_T \quad \text{and} \quad G_{VA} = g_V + g_A \sigma_3$$

L_{int} gives just the sum of the five ~~covari~~ even invariants

$$L_{int} = \sum_i^5 g_i\, \bar{\psi} O_i \psi \qquad (i = S, T, P, A, V).$$

If G_{STP} and G_{VA} commute with σ_3, then we have the most general gauge invariant interaction

$G_{STP} =$

To express this interaction in 4-spinor formalism we ~~need~~ used the identities:

$$\tfrac{1}{2} \bar{\Psi} \Phi) = \mathcal{R}(\bar{\psi}\varphi) - i\,\mathcal{R}\, \bar{\psi} i\gamma_5 \varphi$$

and

$$\bar{\psi}\varphi = \mathcal{R}(\bar{\Psi}\Phi) \pm i\,\mathcal{R}(\bar{\Psi}.\Phi i\sigma_3)$$