New Tensor Wave equations ~~derived from~~ equivalent to a generalized form of Dirac's Equation

~~Tensor Formulation of~~ Quaternion Field Equations

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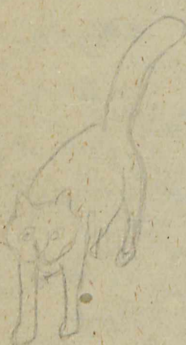
1. Introduction – ~~The the~~ One of the advantages of vector calculus is that it allows one to write ~~the~~ ~~the~~ equations independently of the coordinate system. This invariant form of equations contrasts with the covariant formulation in which the components appear explicitly (this explicit form depends on the co-ordinate system) but in which the relation keeps the same form in various coordinate systems. In special or general relativity the tensor formulation is preferred since the invariant scalar product of four vectors is not sufficient for the invariant formulation of relativistic field equations.



$DA = \Omega$

$[ABC] = G$

$(D,A) = \omega$

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$$Q^\times D \bar{Q} Q \Phi Q^\dagger = \mu Q^\times \Phi^\times \bar{Q} Q L^\times Q^\dagger$$

$$Q^\times D \bar{Q} Q \Phi \bar{Q} = \mu Q^\times \Phi^\times Q^\dagger Q^\times L^\times \bar{Q}$$

$D \Phi = \mu \Phi^\times L^\times$

$L' = Q^\times L^\times Q$

$L' = Q L Q$

$D' \Phi' = \mu \Phi'^\times L'^\times$

$D \Phi = \mu \Phi^\times L^\times$

$$\boxed{\begin{aligned} D' &= Q^\times D \bar{Q} \\ \Phi' &= Q \Phi Q^\dagger \\ L' &= Q^\times L \bar{Q}^\dagger \end{aligned}}$$

resp.

$$\boxed{\begin{aligned} D' &= Q^\times D \bar{Q} \\ \Phi' &= Q \Phi \bar{Q} \\ L' &= Q L Q^\dagger \end{aligned}}$$

$\bar{Q} L'^\times Q^\times = L^\times$

$Q^\dagger L' Q = L$

$L' = Q^\times L \bar{Q}$

$D \Phi Q^\dagger = \mu \Phi^\times \bar{Q} L'^\times Q$

$D \Phi = \mu \Phi^\times \bar{Q} L'^\times Q^\times = \mu \Phi^\times L^\times$

$Q^\times D \bar{Q} \Phi' = \mu \Phi'^\times L'^\times$

$\Phi' = Q \Phi Q^\dagger$

$\cancel{Q^\times} D \Phi Q^\dagger = \mu Q^\times \Phi^\times \bar{Q} L^{\times\prime}$

if $L\bar{L}$ = real L is antihermitian

$L = -L$ $L\bar{L} = 1$

$D \Phi = \mu \Phi^\times L^\times$

$D^\times \Phi^\times = \mu \Phi L$

$D^\times D \Phi = \mu D^\times \Phi^\times L^\times = \mu^2 \Phi L L^\times$

$L L^\times = -1$

$(L\bar{L}) L$

$L\bar{L} = e^{2i\alpha}$

$[N(L)][N(L)]^\times = +1$

$N(L) = e^{i\alpha}$

$L^{-1} = e^{-2i\alpha} \bar{L}$

$L^\times = -L^{-1} = -e^{-2i\alpha} \bar{L}$

$L = -e^{2i\alpha} L^\dagger$

2 - Tensor components of a Quaternion

Let us consider the ~~conf~~ Lorentz transformation

(2.1) $$X' = QXQ^\dagger$$

where X is the hermitian quaternion (2x2 matrix)

(2.2) $$X = I_\nu x^\nu$$

which represents the position vector x^ν and Q is a quaternion with unit norm (or a unimodular 2x2 matrix).

(2.3) ~~If G is a complex~~ $Q\bar{Q} = 1$.

We have already seen that a spinor quaternion Ψ transforms according to the law

(2.4) $$\Psi' = Q\Psi.$$

We now consider complex quaternions with other transformation properties. Let W obey the law

(2.5) $$W' = QWQ^\dagger$$

and G obey the law

(2.6) $$G' = QG\bar{Q}.$$

We can write

(2.7) $$W = U + iV$$

where U and V denote respectively the Hermitian and antihermitian parts of W. It follows from 2.5 that U and V obey separately the law (2.5), i.e.

(2.8) $$U' = QUQ^\dagger$$

(2.9) $$V' = QVQ^\dagger.$$

Hence the components u_ν and v_ν of U and V defined by

(2.10) $$U = I_\nu u^\nu \text{ and } V = I_\nu v^\nu$$

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behave like components of 4-vectors.

On the other hand consider the space-reflection

(2.11) $X'' = \bar{X}^* = \bar{X}$

~~studied in~~ (I, ..) ~~ff~~ Under this improper Lorentz transformation we have

(2.12) $W'' = W^* = U^* - iV^*$

so that

(2.13) $U'' = U^*$

(2.14) $V'' = -V^*$

~~This shows~~ we may conclude that under (2.11), U behaves like a 4-vector and V like a pseudo-vector.

In short, if a complex vector obeying (2.5) is decomposed into its hermitian and antihermitian parts, the hermitian part represents a 4-vector and the antihermitian part a pseudo-vector.

Now a quaternion G obeying (2.6) can be analyzed into its scalar and vectorial parts. We further analyze its scalar part into its real and imaginary parts so that we can write

(2.15) $G = f + ih + \underline{F}$

where f and h are real and $\underline{F}$ is a purely vectorial quaternion. From (2.6) follow the transformation equations

(2.16) $f' = f$

(2.17) $h' = h$

(2.18) $\underline{F}' = Q\underline{F}\bar{Q}$

If under a space reflection we have

(2,19) $G'' = G^{\times}{}'' = f - ih + \underline{F}^{\times}$

which leads to

(2,20) $f'' = f$,

and

(2,21) $h'' = -h$.

These transformation properties show that f is a scalar, h a pseudoscalar and $\underline{F}$ a six vector which can be represented by

(2.22) $$\underline{F} = \frac{1}{2} I_\mu^{\dagger} \bar{I}_\nu F^{\mu\nu}$$

$F^{\mu\nu}$ being the components of an antisymmetrical tensor an I^ν the Hermitian quaternion units (Pauli matrices) defined by ( ).

Hence if a quaternion ~~which transforms~~ transforms according to (2.6), then the real and imaginary parts of its scalar part ~~... scalar~~ are respectively a scalar and a pseudoscalar, while its vectorial part represents a six-vector.

If a 2×2 matrix representation is used for the same quaternion its trace is the complex combination of a scalar and a pseudoscalar while its traceless part represents an antisymmetrical tensor.

In the following complex quaternions which obey the transformation laws (2.4), (2.5) or (2.6) will be respectively called spinor-quaternion, 4-vector-quaternion and 6-vector-quaternion.

3 – Generalized form of Dirac's equation

New equation for the neutrino

We have already seen ( ) that Dirac's equation may be written in quaternion form as

$$(1) \qquad D\Psi = (m\Psi^* + eA\Psi)\underline{e}_3 \; ,$$

Natural units are used. D is the hermitian gradient operator defined by

$$(2) \qquad D = I^\nu \partial_\nu$$

A is the hermitian quaternion associated with the 4-vector A_ν; $\underline{e}_3$ is the third quaternion unit. Ψ is the quaternion which corresponds to the 4-spinor ψ (II) and Ψ^* is the complex conjugate quaternion (represented by the matrix Ψ^c in the 2×2 matrix representation as explained in II). ~~Now~~

~~Let~~ us consider the field free equation

$$(3) \qquad D\Psi = m\Psi^*\underline{e}_3 \, .$$

The quaternion Ψ satisfies the second order equation

$$(4) \qquad D\bar{D}\Psi = \Box\Psi = -m^2\Psi \, .$$

Now the formal generalization of (3) is of the form

$$(5) \qquad D\Phi = (\Phi^* iL)^* = -i\Phi^* L^* \qquad (6)\; L^+ = \bar{L}^\times = -L$$

where L is a constant antihermitian quaternion, that is. We also have

The equation (3) corresponds to the choice $L = i\underline{e}_3 m$

$$(7) \qquad \bar{D}\Phi^* = \Phi\, i L^{\ast}$$

It follows that

$$LL^* = 1$$

$$\Box\Phi = \bar{D}\Psi^* L^* = \Phi LL^* = +\Phi L\bar{L}$$

that is, Φ satisfies an equation: or $\Box\Psi = \Phi L\bar{L}$ sign

L must be antihermitian of spur [illegible]

Comparing with (4) we find the condition

(8) $$L\bar{L}^* = -m^2$$

which expresses that L represents a space-like vector. ~~On the other hand, taking the hermitian conjugate of (5) we obta~~

~~(8) $\Psi^\dagger \mathcal{C} = m L^\dagger \Psi = -m L \Psi$~~

~~with the notations of II and III, From (5) and (8)~~ we obtain

~~$\Psi^\dagger D \Psi + \Psi^\dagger \mathcal{C} \Psi = m\left[(\bar{\Psi}\Psi)^* \Psi \bar{\Psi}\Psi\right]L^*$~~

Now we distinguish two cases

a) $m \neq 0$. Then, ~~one may put~~ according to III, ...) we may express ~~Ψ~~ L in the canonical form

(9) $$L = m C^* i e_3 C^\dagger$$

when C has unit norm.

We now make the transformation

(10) $$\Psi = \Phi \mathbb{C}$$

Then the equation (7) reduces to the standard form (3).

b) $m = 0$. Then L is a null-vector and we may always write

(11) $$L = \lambda C(1 + ie_3)C^\dagger$$

Making the transformation (10) we find

(12) $$D\Psi = \lambda \Psi^*(i + ie_3)$$

If $\lambda = 0$ we find Dirac's equation for a particle of rest mass zero. If $\lambda \neq 0$ we find an alternative equation for a particle of spin $\frac{1}{2}$ and rest mass zero (neutrino). Using the correspondence $(\Psi, \ldots)$ we may write the equation (12) in the 4 spinor form

$$(13) \qquad \gamma^\mu \partial_\mu \psi = \lambda(1+\gamma_5)\psi$$

γ^μ being the 4×4 Dirac-Pauli matrices.

It may be verified directly that (13) leads to the second order equation

$$(14) \qquad \Box \psi = 0$$

assumed to be satisfied by the neutrino.

4) Transformation properties of the generalized Dirac equation

The operator D being a covariant 4-vector, in the Lorentz transformation (7.1) we have

$$(4.1) \qquad D' = Q^\times D \bar{Q}$$

We see that the Dirac equation is invariant if Ψ is a spinor-quaternion, that is

$$(4.2) ~~(4.2)~~ \qquad \Psi' = Q\Psi$$

We then have

$$(4.3) \qquad D'\Psi' = m\Psi'^* \underline{e_3},$$

Now suppose that the ~~arbitrary~~ quaternion

L in ( ) ~~transforms~~ like a 4-vector. If L is contravariant it ~~we have~~ transforms like X (2.1) and we have

(4.4) $L' = Q L Q^{\dagger}$

On the other hand, if L is covariant it transforms like D (4.1) and we have

~~and we see that~~ $L' = Q^* L \bar{Q}$

(4.5) In the first case we obtain from (3.5)

(4.6) $D'\Phi' = Q^* D \bar{Q} \Phi' = -i \Phi'^* Q^* L^* \bar{Q}$

It follows that (3.5) remains invariant if Φ obeys the law

(4.7) $$\Phi' = Q \Phi \bar{Q}$$

that is if Φ is a 6-vector quaternion.

If L is a 4-vector then the quaternion C defined by ( ) is a spinor quaternion, ~~so the~~ which transforms according to the law

(4.8) $$C' = Q C$$

and we have

(4.9) $$\Psi' = \Phi' C' = Q \Phi \bar{Q} Q C = Q \Phi C = Q \Psi$$

so that Ψ is a spinor.

Therefore in the generalized Dirac equation ~~involving a space-like~~ containing a constant 4-vector L which transforms according to (4.4), Φ must be regarded as a 6-vector quaternion.

On the other hand if L is contravariant and transforms according to (4.1) we obtain

(4.10) $D'\Phi' = Q^* D \bar{Q} \Phi' = -i \Phi'^* Q L^* Q^{\dagger}$

or $D \bar{Q} \Phi' Q^* = -i Q^{\dagger} \Phi'^* Q L^*$

so that (3.5) remains invariant if Φ obeys the law

(4.11) $\Phi' = Q \Phi Q^{\dagger}$,

~~~~ which shows that in this case Φ is a 4-vector quaternion. This time the quaternion C obeys the spinor law

(4.12) $C' = Q^* C$

$\bar{Q}\Phi' Q^* = \Phi$

$\Phi' = Q \Phi Q^{\dagger}$
~~~~

it follows that

$$(4.13)\quad \Psi' = \Phi' C' = Q \Phi Q^{\dagger} Q^{\times} C = Q \Psi$$

which means that Ψ in this case is a 4-spinor which obeys the same law (4.9).

which correspond respectively to a contravariant and a covariant space-like vector L

5) Tensor formulations of the generalized Dirac Equation

To express the generalized Dirac equation

$$D \Phi = -i \Phi^{\times} L^{\times}$$

in tensor form ~~we note that~~ we distinguish the contravariant and the covariant cases. In the contravariant case we have a the 6-vector-quaternion Φ which may be analyzed into a scalar, a pseudoscalar and an antisymmetrical tensor according to the formula

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$$(5.1)\quad \Phi = f + ig + \underline{F}$$

where $\underline{F}$ is given by ( ). We obtain

$$(5.2)\quad D(f+ig) + D\underline{F} = -i(f-ig)L^{\times} - i F^{\times} L^{\times},$$

The hermitian part of both sides gives

$$(5.3)\quad Df + \mathcal{H} D\underline{E} = -g L^{\times} - \mathcal{H}\, i(F^{\times} L^{\times})_{D}$$

and the antihermitian part gives

$$(5.4)\quad Dg + -\mathcal{H} D i\underline{F} = -f L^{\times} - \mathcal{H} \underline{F}^{\times} L^{\times}$$

In order to express these equations in tensor form we need a special notation for the space-reflected antisymmetrical tensor. We denote it by $\hat{F}$, so that if $F_{\mu\nu}$ represents an antisym metrical tensor we define $\hat{F}_{\mu\nu}$ by

$$(5.5)\quad \begin{cases} \hat{F}_{om} = -F_{om} = F^{q} \\ \hat{F}_{mn} = F_{mn} \end{cases} \quad (m, n = 1, 2, 3)$$

so that $\hat{F}_{\mu\nu} = F^{\mu\nu}$ and $\hat{L}_{\mu} = L^{\mu}$

We have shown elsewhere ( ) that the dual antisymmetrical matrix is represented by multiplying the corresponding quaternion by i, so that

$$i\underline{F} = \underline{\tilde{F}} = \frac{1}{2} I^{\mu}\bar{I}^{\nu}\tilde{F}_{\mu\nu}.$$

We also use the result (same ref.)

$$\partial(DF) = I^{\mu}_{\nu}\,\partial_{\mu}F^{\mu\nu}$$

as the tensor equivalent of (3) and (5.3) respectively we have therefore

$$(5.6)\quad \begin{cases} \partial_{\nu} f + \partial^{\mu} F_{\mu\nu} = -g\hat{L}_{\nu} + \hat{F}_{\mu\nu}\hat{L}^{\mu} = -g L^{\nu} + F^{\mu\nu}L_{\mu} \\ \partial_{\nu} g - \partial^{\mu}\tilde{F}_{\mu\nu} = +f\hat{L}_{\nu} - \hat{\tilde{F}}_{\mu\nu}\hat{L}^{\mu} = f L^{\nu} + \tilde{F}^{\mu\nu}L_{\mu} \\ \partial_{\nu}L_{\mu} = 0 \qquad L_{\mu}L^{\mu} = -m^{2} \end{cases}$$

If the space reflection operation is not employed then we ~~had to~~ ~~use~~ tensor equations in which covariant and contravariant components are mixed. The ~~equations~~ tensor equations (5.6) which involve 8 real functions and a constant space like vector L_{μ} are entirely equivalent to the generalized Dirac equation ( ) which by the transformation ( ) may be reduced to the ordinary Dirac equation in the absence of field. It may be verified directly that (5.6) lead to the second order equations

$$(5.7)\quad \Box f + m^{2} f = 0 \qquad \Box g + m^{2} g = 0$$

$$\text{and}\quad \Box F_{\mu\nu} + m^{2} F_{\mu\nu} = 0 \quad \text{where}\quad m^{2} = -L_{\mu}L^{\mu}$$

Now, turning to the covariant case Φ is a 4-vector quaternion which can be analyzed into a 4-vector U and a pseudo-4-vector V according to the formula

$$(5.8) \qquad \Phi = U + iV$$

where U and V are given respectively by ( ). Substituting into the generalized Dirac equation we obtain

$$(5.9) \qquad D(U + iV) = -i(\bar{U} - i\bar{V})L^*$$

The scalar part of both sides gives

$$(5.10) \qquad D\cdot U + iD\cdot V = -V^*\cdot L^* - iU^*\cdot L^* = -V\cdot L - iU\cdot L$$

while the vectorial part gives

$$(5.11) \qquad \mathrm{Vec}\, DU + i\,\mathrm{Vec}\, DV = -i\,\mathrm{Vec}\,\hat{U}^* L^* - \mathrm{Vec}\, V^* L^*$$

The equivalent tensor equations are then

$$(5.12) \quad \begin{cases} \partial_\mu U^\mu = -L_\mu V^\mu \\ \partial_\mu V^\mu = -L_\mu U^\mu \\ \partial_\mu U_\nu - \partial_\nu U_\mu + \mathrm{dual}(\partial_\mu V_\nu - \partial_\nu V_\mu) = -(\hat{V}_\mu \hat{L}_\nu - \hat{V}_\nu \hat{L}_\mu) - (\hat{U}_\mu \hat{L}_\nu - \\ \partial_\nu L_\mu = 0 \end{cases}$$

To these we must adjoin again $\partial_\mu L_\nu = 0$ and $L_\mu L^\mu = -m^2$

It is seen that ~~using~~ mixing ~~tensorial~~ tensorial quantities with their dual and space-reflected forms we can write two sets of equations exactly equivalent to the generalized Dirac equation, In ~~each~~ case we have eight unknown functions to determine, a scalar, a pseudo-scalar and an ~~the~~ six components of an antisymmetrical tensor in the contravariant case, ~~and~~ a 4-vector and a pseudo-4-vector in the covariant case. In the

In the neutrino case the constant space-like vector L_μ must be ~~chosen as~~ isotropic, that is

$$L_\mu L^\mu = 0.$$

6 – ~~Generalized~~ Curvilinear coordinates in Space-time; Base Vectors. The foregoing results may be easily ~~generalized to~~ shown to be valid in a general coordinate system if the ordinary differentiation is replaced by covariant differentiation. Consider the transformation

(6.1) $$X' = f(X).$$

The infinitesimal displacement vector assumes the form

(6.2) $$dX' = E_\mu \, dx^\mu$$

where the new base vectors

(6.3) $$E_\mu = \frac{\partial X'}{\partial x_\mu}$$

are represented by ~~[illegible]~~ four hermitian quaternions. Taking the norm of (6.2) we obtain

(6.4) $$N(dX') = ds^2 = E_\mu \cdot \bar{E}_\nu \, dx^\mu dx^\nu = g_{\mu\nu} \, dx^\mu dx^\nu$$

so that the metric tensor is given by

(6.5) $$g_{\mu\nu} = E_\mu \cdot \bar{E}_\nu .$$

Since the line element (6.4) belongs to a flat space ~~we have~~ the Christoffel symbols belonging to the metric (6.5) are given by

(6.6) $$\left\{ {\lambda \atop \mu\nu} \right\} = \bar{E}^\lambda \cdot \partial_\mu E_\nu = \bar{E}^\lambda \cdot \partial_\nu E_\mu .$$

~~[illegible]~~ If covariant differentiation is denoted by the symbol Δ_μ we have

(6.7) $$\Delta_\mu E_\nu = \partial_\mu E_\nu - \left\{ {\lambda \atop \mu\nu} \right\} E_\lambda = 0 .$$

We now consider the quaternions $E_\lambda \bar{E}_\mu$ ~~and $E_\lambda \bar{E}_\mu E_\nu$~~. We can write

(6.8) $$E_\lambda \bar{E}_\mu = g_{\lambda\mu} + \underline{E}_{\lambda\mu}$$

where, according to (6.5) $g_{\mu\nu}$ is ~~the~~ its scalar part, ~~of~~ and the quaternion $\underline{E}_{\lambda\mu}$, antisymmetrical in both its indices denotes its vectorial part. We have

$$(6.9) \quad \underline{E}_{\lambda\mu} = \frac{1}{2}(E_\lambda \bar{E}_\mu - E_\mu \bar{E}_\lambda).$$

~~From the product~~ $\cancel{E}$ Similarly the product $E_\lambda \bar{E}_\mu E_\nu$ can be written as

$$(6.10) \quad E_\lambda \bar{E}_\mu E_\nu = G_{\lambda\mu\nu} + E_{\lambda\mu\nu}$$

where the first term denotes its hermitian part and the second its antihermitian part.

Using the identity (I, 3.1) and (6.5) we find

$$(6.11) \quad G_{\lambda\mu\nu} = \mathcal{H}(E_\lambda \bar{E}_\mu E_\nu) = g_{\lambda\mu} E_\nu + g_{\mu\nu} E_\lambda - g_{\nu\lambda} E_\mu$$

again from the identity (I, 3.3) it follows that the cartesian components of $E_{\lambda\mu\nu}$ are all determinants, hence this antihermitian quaternion is antisymmetrical in all its three indices. ~~From~~ according to the identity (I. 3,8) ~~we also~~ the real quantity

$$(6.12) \quad k_{\kappa\lambda\mu\nu} = -i \bar{E}_\kappa \cdot E_{\lambda\mu\nu}$$

which is equal to the determinant formed by the base vectors E_ν, ~~and~~ is therefore antisymmetrical in all its four indices

~~Let us put~~ introduce the cartesian components of E_ν by the formula

$$(6.13) \quad E_\nu = I^{(\mu)}_{(\mu)} t^{(\mu)}_\nu .$$

Here the bracketted indices refer to the cartesian components and are raised and lowered by means of the metric tensor of special relativity while the unbracketted index is associated with the curvilinear coordinate system (6.1) and is raised or lowered by means of (6.5).

We then have

$$(6.14)\quad k = -k_{0123} = i\,\bar{E}_0 \cdot E_{123} = \mathrm{Det}\left(h^{(\mu)}_{\nu}\right) = \sqrt{-\mathrm{Det}\,g}$$

and the quantity (6.12) is equal to $\mp k$ according as $(\kappa\lambda\mu\nu)$ is an even or odd permutation of (0123).

~~mother words~~ ~~Matt~~ Multiplication of both sides of (6.12) and summation over κ gives

$$(6.15)\quad E_{\lambda\mu\nu} = i\,k_{\kappa\lambda\mu\nu}\,E^{\kappa}.$$

Combining (6.11) and (6.15) we obtain the identity

$$(6.16)\quad E_\lambda E_\mu E_\nu = g_{\lambda\mu} E_\nu + g_{\mu\nu} E_\lambda - g_{\nu\lambda} E_\mu + i\,k_{\kappa\lambda\mu\nu} E^{\kappa}.$$

Another useful identity which is obtained ~~by using~~ from the definition (6.9) by using (6.16) reads

$$(6.17)\quad \underline{E}_{\alpha\beta} \cdot \underline{E}_{\varrho\sigma} = (g_{\alpha\sigma} g_{\beta\varrho} - g_{\alpha\varrho} g_{\beta\sigma}) - i\,k_{\alpha\beta\varrho\sigma}$$

7 - Identities involving dual quantities

Consider the vector a_ε, the antisymmetric tensor ~~of the third rank~~ $b_{\varrho\sigma}$ and the antisymmetric tensor of the third rank $c_{\varrho\sigma\tau}$. Duality is defined by means of the totally antisymmetrical tensor of the fourth rank $k_{\alpha\beta\gamma\varepsilon}$ introduced in the preceding section. Thus

(7.1) $\text{dual } a_\varepsilon = \tilde{a}^{\alpha\beta\gamma} = k^{\alpha\beta\gamma\varepsilon} a_\varepsilon$

(7.2) $\text{dual } b_{\rho\sigma} = \tilde{b}^{\alpha\beta} = \frac{1}{2!} k^{\alpha\beta\rho\sigma} b_{\rho\sigma}$

(7.3) $\text{dual } c_{\rho\sigma\tau} = \tilde{c}^{\alpha} = \frac{1}{3!} k^{\alpha\rho\sigma\tau} c_{\rho\sigma\tau}$.

where
(7.4) $k^{0123} = k^{-1}$.

Introducing the generalized Kronecker symbols (cf. Brand [ ], pp. 353, 369) we can write

(7.5) $k_{\kappa\lambda\mu\nu} = -k\, \delta^{0123}_{\kappa\lambda\mu\nu}$

and
(7.6) $k^{\alpha\beta\lambda\mu} = k^{-1} \delta^{\alpha\beta\lambda\mu}_{0123}$

so that we have
(7.7) $k^{\alpha\lambda\mu\nu} k_{\beta\lambda\mu\nu} = -3!\, \delta^\alpha_\beta$

and
(7.8) $k_{\rho\sigma\alpha\beta} k^{\lambda\mu\alpha\beta} = -\delta^{\lambda\mu\alpha\beta}_{\rho\sigma\alpha\beta} = -2!\, \delta^{\lambda\mu}_{\rho\sigma}$

where $\delta^{\lambda\mu}_{\rho\sigma} = (\delta^\lambda_\rho \delta^\mu_\sigma - \delta^\lambda_\sigma \delta^\mu_\rho)$.

Hence we obtain
(7.9) $\text{dual dual } a_\varepsilon = \text{dual } \tilde{a}^{\alpha\beta\sigma} = a_\varepsilon$

(7.10) $\text{dual dual } b_{\rho\sigma} = -b_{\rho\sigma}$.

When applied to the bases vectors and the antisymmetrical quaternions derived from them ~~we obtain~~ the same definitions give

$$(\qquad \text{dual } E_\epsilon = k^{\alpha\beta\gamma\epsilon} E_\epsilon = -i E^{\alpha\beta\gamma}$$

so that we have

$$(7.11) \qquad \tilde{E}^{\alpha\beta\gamma} = -i E^{\alpha\beta\gamma}$$

or

$$(7.11') \qquad E_{\alpha\beta\gamma} = i \tilde{E}_{\alpha\beta\gamma} .$$

Now we want to prove that

$$(7.12) \qquad \underline{E}_{\alpha\beta} = i \tilde{E}_{\alpha\beta} .$$

To this end, consider an arbitrary real antisymmetrical tensor $f^{\alpha\beta}$ and form the purely vectorial quaternion

$$(7.13) \qquad \underline{F} = \tfrac{1}{2} \underline{E}_{\alpha\beta} f^{\alpha\beta}$$

We have, using the identity (6.17) and the definition (7.2)

$$(7.14) \qquad \underline{E}_{\rho\sigma} \cdot \underline{F} = \tfrac{1}{2} (\underline{E}_{\alpha\beta} \cdot E_{\rho\sigma}) f^{\alpha\beta} = (f_{\sigma\rho} - i \tilde{f}_{\rho\sigma})$$

Hence, using (6.17) we can write

$$(7.15) \qquad \underline{F} = \tfrac{1}{2} \underline{E}_{\alpha\beta} f^{\alpha\beta} = \tfrac{1}{4} \underline{E}^{\rho\sigma} \mathcal{R} \{ \underline{E}_{\rho\sigma} \cdot \underline{E}_{\alpha\beta} f^{\alpha\beta} \}$$

where $\mathcal{R}$ ~~stands~~ for "Real Part".

Similarly the quaternion $i\underline{F}$ is represented by

$$(7.16) \qquad \tfrac{1}{2} i \underline{E}_{\alpha\beta} f^{\alpha\beta} = \underline{E}^{\rho\sigma} \mathcal{R} \{ \underline{E}_{\rho\sigma} \cdot i \underline{E}_{\alpha\beta} f^{\alpha\beta} \} = \tfrac{1}{4} \underline{E}^{\rho\sigma} k_{\rho\sigma\alpha\beta} f^{\alpha\beta}$$

or, ~~on using~~ Making use of the definition

$$\tilde{E}_{\alpha\beta} = \frac{1}{2!} k_{\alpha\beta\rho\sigma} \underline{E}^{\rho\sigma}$$

(7.16) takes the form

(7.17) $$i\underline{F} = \frac{1}{2} \tilde{E}_{\alpha\beta} f^{\alpha\beta}$$

~~so that~~ combination of (7.13) and (7.17) shows that, for an arbitrary tensor $f^{\alpha\beta}$ we must have

$$\left(\underline{E}_{\alpha\beta} - i\tilde{E}_{\alpha\beta}\right) f^{\alpha\beta} = 0$$

and this proves (7.12)

§ 8. Tensor Components of an invariant quaternion

We now come to our main result:

Any invariant quaternion Q may be expressed in one of the forms

(8.1) $$Q = \underline{E}_\alpha a^\alpha + \frac{1}{3!} E^{\lambda\mu\nu} b_{\lambda\mu\nu}$$

or

(8.2) $$Q = u + \frac{i}{4!} k^{\lambda\mu\nu\varepsilon} v_{\lambda\mu\nu\varepsilon} + \frac{1}{2}\underline{E}^{\rho\sigma} f_{\rho\sigma}$$

where a^α is a real 4-vector, ~~$b_{\lambda\mu\nu}$ a real antisymmetrical tensor of rank 3~~, u an real invariant scalar; ~~$v_{\lambda\mu\nu\varepsilon}$ a real antisymmetrical tensor of rank 4 and $f_{\rho\sigma}$ a~~ and $f_{\rho\sigma}$, $b_{\lambda\mu\nu}$, $v_{\lambda\mu\nu\varepsilon}$ real antisymmetrical tensors of rank 2, 3 and 4 respectively.

To prove (8.1) we multiply both sides scalarly by E^ρ. We obtain

$$\overline{E}^{\rho} \cdot Q = a^{\rho} - \frac{i}{3!} k^{\rho\lambda\mu\nu} b_{\lambda\mu\nu} = a^{\rho} - i\tilde{b}^{\rho}.$$

Hence a^{ρ} and $\tilde{b}^{\rho}$ (hence its dual $b_{\lambda\mu\nu}$) are determined uniquely by the relations

$$(8.3) \quad a^{\rho} = \mathcal{R}(\overline{E}^{\rho} \cdot Q)$$

$$(8.4) \quad \tilde{b}^{\rho} = \mathcal{R}(i\overline{E}^{\rho} \cdot Q);$$

The dual of an antisymmetrical tensor of rank 3 is a pseudo-vector. Therefore any invariant complex quaternion may be regarded as having vector and pseudo vector components corresponding respectively to its hermitian and anti-hermitian parts.

In the relation (8.2) the dual quantity

$$(8.5) \quad \tilde{v} = \frac{1}{4!} k^{\lambda\mu\nu\varepsilon} v_{\lambda\mu\nu\varepsilon}$$

of the antisymmetrical tensor of the fourth rank is a pseudo-scalar. u and $\tilde{v}$ are determined uniquely by

$$(8.6) \quad u + i\tilde{v} = \mathcal{S}Q = \tfrac{1}{2}(Q + \overline{Q}).$$

where $\mathcal{S}$ denotes the scalar part.

From (7.14) we may find $f_{\alpha\beta}$ by

$$(8.7) \quad f_{\alpha\beta} = -\mathcal{R}(\underline{E}_{\alpha\beta} \cdot \underline{F})$$

where

$$\underline{F} = \mathcal{V}Q = \tfrac{1}{2}(Q - \overline{Q})$$

$\mathcal{V}$ denoting the vectorial part

It follows that any invariant quaternion may also be regarded as having an invariant scalar component, a pseudo-invariant scalar component and real antisymmetrical tensor components. The first two are associated with its scalar part and the six-vector with its vectorial part.

Now, using the representation (8.1) for Φ in the generalized Dirac equation (5.5) we find the set (5.12) with the covariant differentiation replacing ordinary differentiation. Similarly using the representation (8.2) we find the set (5.6). This shows that the tensor equations (5.6) and (5.12) are indeed the tensor equivalents of the generalized Dirac equation with Δ_μ replacing ∂_μ.